Online Appendix to "Optimal Monetary Policy in Small Open Economies: Producer-Currency Pricing"

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Appendix A. Risk Sharing

The household in country i maximizes lifetime utility (1), subject to the household budget constraint (2) and the intertemporal asset constraint

$$\mathbb{E}\left\{\sum_{t=1}^{\infty} q_t \mathcal{D}_{it}\right\} = 0. \tag{A.1}$$

D_{it} denotes nominal state-contingent portfolio payments in units of imported consumption baskets in period t, while q_t is the price of the portfolio in period 0 (when all trading occurs), where q_t does not depend on country i. The household in period 0 cares about the relative price of contingent claims across all states. The intertemporal asset constraint stipulates that all period 0 transactions must be balanced: payment for claims issued must equal payment for claims received. The household Lagrangian is

$$\begin{aligned}\mathcal{L}_{it} = \sum_{t=1}^{\infty} \beta^t \mathbb{E}\Bigg\{\frac{C_{it}^{1-\sigma}}{1-\sigma} - \frac{N_{it}^{1+\varphi}}{1+\varphi} + \frac{\lambda_{it}}{P_{it}}\Big[W_{it}N_{it} + P_{F,it}\mathcal{D}_{it} - P_{it}C_{it}\Big]\Bigg\} \\ - \lambda_0 \sum_{t=1}^{\infty} \mathbb{E}\big\{q_t \mathcal{D}_{it}\big\}. \end{aligned} \tag{A.2}$$

Note that the Lagrange multiplier λ_0 is common across countries only under the assumption of ex ante symmetry. The first-order conditions with respect to nominal state-contingent portfolio payments D_{it} and consumption C_{it} are

$$\frac{\partial \mathcal{L}_{it}}{\partial D_{it}} = -\lambda_0 q_t + \beta^t \lambda_{it} \frac{P_{F,it}}{P_{it}} = 0, \tag{A.3}$$

$$\frac{\partial \mathcal{L}_{it}}{\partial C_{it}} = \beta^t C_{it}^{-\sigma} - \beta^t \lambda_{it} = 0. \tag{A.4}$$

Combining (A.3) and (A.4) gives the price of the portfolio q_t:

$$q_t = \frac{\beta^t}{\lambda_0} \frac{P_{F,it}}{P_{it}} C_{it}^{-\sigma} = \frac{\beta^t}{\lambda_0} \frac{C_{it}^{-\sigma}}{\tilde{P}_{it}}. \tag{A.5}$$

Substituting (A.5) into (A.1), we can express the intertemporal asset constraint as

$$\mathbb{E}\left\{ \sum_{t=1}^{\infty} \beta^t \frac{C_{it}^{-\sigma}}{\tilde{P}_{it}} \mathcal{D}_{it} \right\} = 0, \tag{A.6}$$

which corresponds to equation (28) in the text.

Since the price of the portfolio q_t and the Lagrange multiplier λ_0 do not depend on country i, the risk-sharing condition in complete markets is

$$\frac{C_{it}^{-\sigma}}{P_{it}} = \frac{C_{jt}^{-\sigma}}{P_{jt}} = \frac{q_t \lambda_0}{\beta^t} \quad \forall \quad i, j.$$

Therefore, $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{\sigma}}$, where A_t is known unconditionally, as there is no aggregate uncertainty. Since total portfolio payments globally add up to zero in each period,

$$\mathbb{E}\{\mathcal{D}_{it}\} = \mathbb{E}\big\{\tilde{P}_{it} C_{it} - Y_{it} \tilde{P}_{H,it}\big\} = 0,$$

$$\Rightarrow \mathbb{E}\big\{\tilde{P}_{it} C_{it}\big\} = \mathbb{E}\big\{Y_{it} \tilde{P}_{H,it}\big\}. \tag{A.7}$$

Now substitute the expression $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{\sigma}}$ for consumption in (A.7)

$$A_t = \frac{\mathbb{E}\big\{Y_{it} \tilde{P}_{H,it}\big\}}{\mathbb{E}\big\{\tilde{P}_{it}^{\frac{\sigma-1}{\sigma}}\big\}}, \tag{A.8}$$

and solve for C_{it} using $C_{it} = A_t \tilde{P}_{it}^{-\frac{1}{\sigma}}$:

$$C_{it} = \frac{\mathbb{E}\{Y_{it}\tilde{P}_{H,it}\}}{\mathbb{E}\{\tilde{P}_{it}^{\frac{\sigma-1}{\sigma}}\}}\tilde{P}_{it}^{-\frac{1}{\sigma}}. \tag{A.9}$$

Appendix B. Propositions and Proofs

B.1 Proof of Proposition 1

B.1.1 Setting Up the Langrangian

In complete markets, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 1$ and $\mathbb{1}_{CM} = 1$. Thus, we can formulate a Lagrangian:

$$\begin{aligned}\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \mathbb{E}\Bigg[&\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_1(E_{t-1}C_t^{1-\sigma} - E_{t-1}N_t^{1+\varphi})\\ &+ \lambda_{2,t}((1-\alpha)\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta} + \alpha\tilde{P}_{H,t}^{-\gamma}\mathbb{E}[C_t\tilde{P}_t^{\eta}] - Z_tN_t)\\ &+ \lambda_{3,t}((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha - \tilde{P}_t^{1-\eta})\\ &+ \lambda_{4,t}\left(\mathbb{E}\{Z_tN_t\tilde{P}_{H,t}\} - C_t\tilde{P}_t^{\frac{1}{\sigma}}\mathbb{E}[\tilde{P}_t^{\frac{\sigma-1}{\sigma}}]\right)\Bigg].\end{aligned} \tag{B.1}$$

The first-order conditions for the problem with respect to consumption C_t, labor N_t, the terms of trade $\tilde{P}_{H,t}$, and the real exchange rate $\tilde{P}_t$ are given below:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial C_t} = &\, C_t^{-\sigma} + \lambda_1(1-\sigma)C_t^{-\sigma} + \lambda_{2,t}(1-\alpha)\tilde{P}_{H,t}^{-\eta}\tilde{P}_t^{\eta}\\ &+ \alpha E_{t-1}(\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma})\tilde{P}_t^{\eta}, \quad -\lambda_{4,t}\tilde{P}_t^{\frac{1}{\sigma}}\mathbb{E}[\tilde{P}_t^{\frac{\sigma-1}{\sigma}}] = 0,\end{aligned} \tag{B.2}$$

$$\frac{\partial\mathcal{L}}{\partial N_t} = -N_t^{\varphi} - \lambda_1(1+\varphi)N_t^{\varphi} - \lambda_{2,t}Z_t + \mathbb{E}(\lambda_{4,t})Z_t\tilde{P}_{H,t} = 0, \tag{B.3}$$

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial \tilde{P}_{H,t}} = &-\eta(1-\alpha)\lambda_{2,t}\tilde{P}_{H,t}^{-\eta-1}C_t\tilde{P}_t^{\eta} - \gamma\alpha\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma-1}E_{t-1}[C_t\tilde{P}_t^{\eta}]\\ &+ \lambda_{3,t}(1-\alpha)(1-\eta)\tilde{P}_{H,t}^{-\eta} + \mathbb{E}(\lambda_{4,t})Z_tN_t = 0,\end{aligned} \tag{B.4}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = (1-\alpha)\eta\lambda_{2,t}\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta-1} + \alpha\eta E_{t-1}(\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma})C_t\tilde{P}_t^{\eta-1} - (1-\eta)\lambda_{3,t}\tilde{P}_t^{-\eta} - \lambda_{4,t}\frac{1}{\sigma}\tilde{P}_t^{\frac{1}{\sigma}-1}C_t\mathbb{E}\tilde{P}_t^{1-\frac{1}{\sigma}} - \left(1-\frac{1}{\sigma}\right)\mathbb{E}\left[\lambda_{4,t}C_t\tilde{P}_t^{\frac{1}{\sigma}}\right]\tilde{P}_t^{-\frac{1}{\sigma}} = 0. \tag{B.5}$$

The first-order conditions (B.2)–(B.5), constraints (35)–(38), cooperation indicator $\mathbb{1}_{CP} = 1$, complete markets indicator $\mathbb{1}_{CM} = 1$, and exogenous shock dynamics (33) describe the full nonlinear dynamics of the system. To obtain an analytical expression, we have to consider the behavior of the model in the steady state.

B.1.2 Steady State

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), goods and asset market clearing (36)–(38) in the steady state allows us to show that $C = N = \tilde{P}_H = \tilde{P} = 1$. Substituting these values into the first-order conditions (B.2)–(B.5) yields the following steady-state relationships:

$$\frac{\partial \mathcal{L}}{\partial C} = 1 + \lambda_1(1-\sigma) + \lambda_2(1-\alpha) + \alpha\lambda_2 - \lambda_4 = 0, \tag{B.6}$$

$$\frac{\partial \mathcal{L}}{\partial N} = -1 - \lambda_1(1+\varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.7}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_H} = -\eta(1-\alpha)\lambda_2 - \gamma\alpha\lambda_2 + \lambda_3(1-\alpha)(1-\eta) + \lambda_4 = 0, \tag{B.8}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}} = (1-\alpha)\eta\lambda_2 + \alpha\eta\lambda_2 - (1-\eta)\lambda_3 - \lambda_4\frac{1}{\sigma} - (1-\frac{1}{\sigma})\lambda_4 = 0. \tag{B.9}$$

We solve the system (B.6)–(B.9) and obtain $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [0, \frac{1}{\gamma-1}, \frac{\gamma-\eta}{(\eta-1)(\gamma-1)}, \frac{\gamma}{\gamma-1}]$.

B.1.3 Log-Linearization

We log-linearize the first-order conditions (B.2)–(B.5) around the deterministic steady state and obtain

$$0 = -\sigma(1+\lambda_1(1-\sigma))\hat{C}_t + \lambda_2(1-\alpha)(\hat{\lambda}_{2,t} - \eta\hat{\hat{P}}_{H,t} + \eta\hat{P}_t) + \alpha\lambda_2\eta\hat{P}_t - \lambda_4(\hat{\lambda}_{4,t} + \frac{1}{\sigma}\hat{\hat{P}}_t), \quad \text{(B.10)}$$

$$0 = -(1+\lambda_1(1+\varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{Z}_t + \hat{\hat{P}}_{H,t}), \quad \text{(B.11)}$$

$$0 = -\eta(1-\alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta+1)\hat{\hat{P}}_{H,t} + \hat{C}_t + \eta\hat{\hat{P}}_t) - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma+1)\hat{\hat{P}}_{H,t}) + \lambda_3(1-\alpha)(1-\eta)(\hat{\lambda}_{3,t} - \eta\hat{\hat{P}}_{H,t}) + \lambda_4(\hat{Z}_t + \hat{N}_t), \quad \text{(B.12)}$$

$$0 = (1-\alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{\hat{P}}_{H,t} + \hat{C}_t + (\eta-1)\hat{\hat{P}}_t) + \alpha\eta\lambda_2(\hat{C}_t + (\eta-1)\hat{\hat{P}}_t) - (1-\eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{\hat{P}}_t) - \lambda_4\frac{1}{\sigma}\left(\hat{\lambda}_{4,t} + \left(\frac{1}{\sigma} - 1\right)\hat{\hat{P}}_t + \hat{C}_t\right) + \left(1 - \frac{1}{\sigma}\right)\frac{1}{\sigma}\lambda_4\hat{\hat{P}}_t. \quad \text{(B.13)}$$

Now we log-linearize the constraints (35)–(38) after setting the value for the indicators $\mathbb{1}_{CP} = 1$, $\mathbb{1}_{CM} = 1$:

$$0 = -\hat{Z}_t - \hat{N}_t + (1-\alpha)(-\eta\hat{\hat{P}}_{H,t} + \hat{C}_t + \eta\hat{\hat{P}}_t) - \alpha\gamma\hat{\hat{P}}_{H,t}, \quad \text{(B.14)}$$

$$0 = (1-\alpha)\hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t, \quad \text{(B.15)}$$

$$0 = \hat{C}_t + \frac{1}{\sigma}\hat{\hat{P}}_t. \quad \text{(B.16)}$$

We can express the system of linear equations consisting of (B.10)–(B.16) as

$$A(\theta)X_t + b(\theta)Z_t = 0, \quad \text{(B.17)}$$

where $X_t = [\hat{C}_t, \hat{N}_t, \hat{\hat{P}}_{H,t}, \hat{\hat{P}}_t, \hat{\lambda}_{2,t}, \hat{\lambda}_{3,t}, \hat{\lambda}_{4,t}]'$, A is a 6×6 matrix, and b is a 6×1 vector. After plugging in the values for $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, we can express coefficients in A and b as functions of the model parameters $\theta = [\sigma, \varphi, \alpha, \eta, \gamma]$. Finally, the endogenous variables X_t can be expressed as a function of the parameter vector θ and the exogenous variable Z_t:

$$X_t = -A(\theta)^{-1}b(\theta)Z_t. \tag{B.18a}$$

We can also express the markup as a function of parameters θ and the technology shock Z_t by log-linearizing (25):

$$\hat{\mu}_t = \hat{Z}_t + \hat{\tilde{P}}_{H,t} - \hat{\tilde{P}}_t - \varphi\hat{N}_t - \sigma\hat{C}_t. \tag{B.19}$$

Plugging the solution from (B.18) into (B.19) yields the following expression for the log-linear markup:

$$\hat{\mu}_t = 0. \tag{B.20}$$

We refer the reader to Dmitriev and Hoddenbagh (2019) for proof that the flexible-price allocation (they consider the flexible-wage allocation, but the algebra is identical) under complete markets coincides with the social planner allocation.

B.2 Proof of Proposition 2

B.2.1 Setting Up the Lagrangian

Under financial autarky, cooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 1$ and $\mathbb{1}_{CM} = 0$. The Lagrangian is

$$\begin{aligned}\mathcal{L} = \sum_{t=1}^{\infty}\beta^t\mathbb{E}\Bigg[&\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_1(E_{t-1}C_t^{1-\sigma} - E_{t-1}N_t^{1+\varphi})\\ &+ \lambda_{2,t}((1-\alpha)\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta} + \alpha\tilde{P}_{H,t}^{-\gamma}\mathbb{E}_{t-1}[C_t\tilde{P}_t^{\eta}] - Z_tN_t)\\ &+ \lambda_{3,t}((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha - \tilde{P}_t^{1-\eta}) + \lambda_{4,t}\left(Z_tN_t\tilde{P}_{H,t} - C_t\tilde{P}_t\right)\Bigg].\end{aligned} \tag{B.21}$$

The first-order conditions with respect to consumption C_t, labor N_t, terms of trade $\tilde{P}_{H,t}$, and real exchange rate $\tilde{P}_t$ are

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial C_t} = {}& C_t^{-\sigma} + \lambda_1(1-\sigma)C_t^{-\sigma} + \lambda_{2,t}(1-\alpha)\tilde{P}_{H,t}^{-\eta}\tilde{P}_t^{\eta}\\ &+ \alpha E_{t-1}(\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma})\tilde{P}_t^{\eta} - \lambda_{4,t}\tilde{P}_t = 0,\end{aligned} \tag{B.22}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -N_t^{\varphi} - \lambda_1(1+\varphi)N_t^{\varphi} - \lambda_{2,t} Z_t + \lambda_{4,t} Z_t \tilde{P}_{H,t} = 0, \quad \text{(B.23)}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_{H,t}} = -\eta(1-\alpha)\lambda_{2,t}\tilde{P}_{H,t}^{-\eta-1} C_t \tilde{P}_t^{\eta} - \gamma\alpha\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma-1} E_{t-1}[C_t \tilde{P}_t^{\eta}]$$

$$+ \lambda_{3,t}(1-\alpha)(1-\eta)\tilde{P}_{H,t}^{-\eta} + \lambda_{4,t} Z_t N_t = 0, \quad \text{(B.24)}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = (1-\alpha)\eta\lambda_{2,t}\tilde{P}_{H,t}^{-\eta} C_t \tilde{P}_t^{\eta-1} + \alpha\eta E_{t-1}(\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma}) C_t \tilde{P}_t^{\eta-1}$$

$$- (1-\eta)\lambda_{3,t}\tilde{P}_t^{-\eta} - \lambda_{4,t} C_t = 0. \quad \text{(B.25)}$$

The first-order conditions (B.22)–(B.25), constraints (35)–(38), cooperation indicator $\mathbb{1}_{CP} = 1$, complete markets indicator $\mathbb{1}_{CM} = 0$, and exogenous shock dynamics (33) describe the full nonlinear dynamics of the system. To obtain an analytical expression, we first need to solve for the steady state.

B.2.2 Steady State, Financial Autarky, Cooperative

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), and goods and asset market clearing (36)–(38) in the steady state allows us to show that $C = N = \tilde{P}_H = \tilde{P} = 1$. Substituting these values into the first-order conditions (B.22)–(B.25) yields the following steady-state relationships:

$$\frac{\partial \mathcal{L}}{\partial C} = 1 + \lambda_1(1-\sigma) + \lambda_2(1-\alpha) + \alpha\lambda_2 - \lambda_4 = 0, \quad \text{(B.26)}$$

$$\frac{\partial \mathcal{L}}{\partial N} = -1 - \lambda_1(1+\varphi) - \lambda_2 + \lambda_4 = 0, \quad \text{(B.27)}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_H} = -\eta(1-\alpha)\lambda_2 - \gamma\alpha\lambda_2 + \lambda_3(1-\alpha)(1-\eta) + \lambda_4 = 0, \quad \text{(B.28)}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}} = (1-\alpha)\eta\lambda_2 + \alpha\eta\lambda_2 - (1-\eta)\lambda_3 - \lambda_4 = 0. \quad \text{(B.29)}$$

We solve the system (B.26)–(B.29) and obtain $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [0, \frac{1}{\gamma-1}, \frac{\gamma-\eta}{(\eta-1)(\gamma-1)}, \frac{\gamma}{\gamma-1}]$. The steady state fully coincides with the steady state for cooperative policy under complete markets.

B.2.3 Log-Linearization

We log-linearize the first-order conditions (B.22)–(B.25) around the deterministic steady state and obtain

$$
\begin{aligned}
0 &= -\sigma(1+\lambda_1(1-\sigma))\hat{C}_t + \lambda_2(1-\alpha)(\hat{\lambda}_{2,t} - \eta\hat{\tilde{P}}_{H,t} + \eta\hat{\tilde{P}}_t) \\
&\quad + \alpha\lambda_2\eta\hat{\tilde{P}}_t - \lambda_4(\hat{\lambda}_{4,t} + \hat{\tilde{P}}_t), \qquad \text{(B.30)} \\
0 &= -(1+\lambda_1(1+\varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{\tilde{P}}_{H,t}), \qquad \text{(B.31)} \\
0 &= -\eta(1-\alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta+1)\hat{\tilde{P}}_{H,t} + \hat{C}_t + \eta\hat{\tilde{P}}_t) \\
&\quad - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma+1)\hat{\tilde{P}}_{H,t}) \\
&\quad + \lambda_3(1-\alpha)(1-\eta)(\hat{\lambda}_{3,t} - \eta\hat{\tilde{P}}_{H,t}) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{N}_t), \qquad \text{(B.32)} \\
0 &= (1-\alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{\tilde{P}}_{H,t} + \hat{C}_t + (\eta-1)\hat{\tilde{P}}_t) \\
&\quad + \alpha\eta\lambda_2(\hat{C}_t + (\eta-1)\hat{\tilde{P}}_t) - (1-\eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{\tilde{P}}_t) \\
&\quad - \lambda_4(\hat{\lambda}_{4,t} + \hat{C}_t). \qquad \text{(B.33)}
\end{aligned}
$$

Now we log-linearize the constraints (35)–(38) after setting the value for the indicators $\mathbb{1}_{CP} = 1$, $\mathbb{1}_{CM} = 0$:

$$0 = -\hat{Z}_t - \hat{N}_t + (1-\alpha)(-\eta\hat{\tilde{P}}_{H,t} + \hat{C}_t + \eta\hat{\tilde{P}}_t) - \alpha\gamma\hat{\tilde{P}}_{H,t}, \qquad \text{(B.34)}$$

$$0 = (1-\alpha)\hat{\tilde{P}}_{H,t} - \hat{\tilde{P}}_t, \qquad \text{(B.35)}$$

$$0 = -\hat{C}_t - \hat{\tilde{P}}_t + \hat{Z}_t + \hat{N}_t + \hat{P}_{H,t}. \qquad \text{(B.36)}$$

We can express the system of linear equations consisting of (B.30)–(B.36) as

$$A(\theta)X_t + b(\theta)Z_t = 0, \qquad \text{(B.37)}$$

where $X_t = [\hat{C}_t, \hat{N}_t, \hat{\tilde{P}}_{H,t}, \hat{\tilde{P}}_t, \hat{\lambda}_{2,t}, \hat{\lambda}_{3,t}, \hat{\lambda}_{4,t}]'$, A is a 6×6 matrix, and b is a 6×1 vector. After plugging in the values for $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, we can express coefficients in A and b as functions of the model parameters $\theta = [\sigma, \varphi, \alpha, \eta, \gamma]$. Finally, the endogenous variables X_t can be expressed as a function of the parameter vector θ and the exogenous variable Z_t:

$$X_t = -A(\theta)^{-1} b(\theta) Z_t. \tag{B.38}$$

We can also express the markup as function of parameters θ and the technology shock Z_t by log-linearizing (25):

$$\mu_t = \hat{Z}_t + \hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \tag{B.39}$$

B.2.4 Analytical Expression

Plugging the solution from (B.38) into (B.39) yields the following expression for the log-linear markup:

$$\hat{\mu}_t = \frac{G_2(\theta)}{F_2(\theta)} \hat{Z}_t, \tag{B.40}$$

where

$$G_2(\theta) = -\alpha(1+\varphi)((1-\alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)), \tag{B.41}$$

$$\begin{aligned} F_2(\theta) = {} & \varphi + \sigma + \alpha\eta + \alpha\gamma - 2\eta\sigma + \eta^2\sigma + \gamma^2\sigma + \alpha^2\eta^2\sigma \\ & - 2\gamma\sigma - \alpha^2\eta - 2\alpha\eta^2\sigma - 2\alpha\eta\gamma\sigma + 2\eta\gamma\sigma + 2\alpha\eta\sigma \\ & - 2\alpha\varphi - 2\eta\varphi - 2\gamma\varphi + \alpha^2\varphi + \eta^2\varphi + \gamma^2\varphi + \alpha^2\eta^2\varphi + 4\alpha\eta\varphi \\ & + 2\alpha\gamma\varphi + 2\eta\gamma\varphi - 2\alpha\eta^2\varphi - 2\alpha^2\eta\varphi - 2\alpha\eta\gamma\varphi. \end{aligned} \tag{B.42}$$

We can express output $\hat{Y}_t = \hat{N}_t + \hat{Z}_t$ as a function of the technology shock using equation (B.38):

$$\hat{Y}_t = \frac{G_{3,Y}}{F_{3,Y}} \hat{Z}_t, \tag{B.43}$$

where

$$\begin{aligned} G_{3,Y} = {} & \varphi - 2\eta - 2\gamma - 2\alpha + 4\alpha\eta + 2\alpha\gamma + 2\eta\gamma - 2\alpha\varphi - 2\eta\varphi \\ & - 2\gamma\varphi - 2\alpha\eta^2 - 2\alpha^2\eta + \alpha^2\varphi + \eta^2\varphi \\ & + \gamma^2\varphi + \alpha^2 + \eta^2 + \gamma^2 + \alpha^2\eta^2 + \alpha^2\eta^2\varphi - 2\alpha\eta\gamma + 4\alpha\eta\varphi \\ & + 2\alpha\gamma\varphi + 2\eta\gamma\varphi - 2\alpha\eta^2\varphi - 2\alpha^2\eta\varphi - 2\alpha\eta\gamma\varphi + 1, \end{aligned} \tag{B.44}$$

$$
\begin{aligned}
F_{3,Y} = {} & \varphi + \sigma + \alpha\eta + \alpha\gamma - 2\alpha\varphi - 2\eta\varphi - 2\gamma\varphi - 2\eta\sigma - 2\gamma\sigma \\
& - \alpha^2\eta + \alpha^2\varphi + \eta^2\varphi + \gamma^2\varphi + \eta^2\sigma + \gamma^2\sigma + \alpha^2\eta^2\varphi \\
& + \alpha^2\eta^2\sigma + 4\alpha\eta\varphi + 2\alpha\gamma\varphi + 2\alpha\eta\sigma + 2\eta\gamma\varphi + 2\eta\gamma\sigma - 2\alpha\eta^2\varphi \\
& - 2\alpha^2\eta\varphi - 2\alpha\eta^2\sigma - 2\alpha\eta\gamma\sigma - 2\alpha\eta\gamma\varphi.
\end{aligned} \tag{B.45}
$$

Given the solutions for the markup and output in terms of technology shocks allows us to obtain their ratio and express it below:

$$
\hat{\mu}_t = -\alpha \frac{(1-\alpha)(\eta\sigma - 1) + \sigma(\gamma - 1)}{(1-\alpha)^2(\eta-1)^2 + (\gamma-1)^2 + 2\eta\gamma(1-\alpha) + 2\alpha\gamma - 1} \hat{Y}_t. \tag{B.46}
$$

B.3 Proof of Proposition 3

B.3.1 Setting Up the Lagrangian

Under financial autarky, noncooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 0$ and $\mathbb{1}_{CM} = 0$. The Lagrangian is

$$
\begin{aligned}
\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \mathbb{E} \Bigg[& \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_1 (E_{t-1} C_t^{1-\sigma} - E_{t-1} N_t^{1+\varphi}) \\
& + \lambda_{2,t}((1-\alpha)\tilde{P}_{H,t}^{-\eta} C_t \tilde{P}_t^{\eta} + \alpha \tilde{P}_{H,t}^{-\gamma} C_F - Z_t N_t) \\
& + \lambda_{3,t}((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha - \tilde{P}_t^{1-\eta}) \\
& + \lambda_{4,t}\left(Z_t N_t \tilde{P}_{H,t} - C_t \tilde{P}_t \right) \Bigg].
\end{aligned} \tag{B.47}
$$

The first-order conditions with respect to consumption C_t, labor N_t, the terms of trade $\tilde{P}_{H,t}$, and the real exchange rate $\tilde{P}_t$ are given below:

$$
\frac{\partial \mathcal{L}}{\partial C_t} = C_t^{-\sigma} + \lambda_1(1-\sigma)C_t^{-\sigma} + \lambda_{2,t}(1-\alpha)\tilde{P}_{H,t}^{-\eta}\tilde{P}_t^{\eta} - \lambda_{4,t}\tilde{P}_t = 0, \tag{B.48}
$$

$$
\frac{\partial \mathcal{L}}{\partial N_t} = -N_t^{\varphi} - \lambda_1(1+\varphi)N_t^{\varphi} - \lambda_{2,t}Z_t + \lambda_{4,t}Z_t\tilde{P}_{H,t} = 0, \tag{B.49}
$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_{H,t}} = -\eta(1-\alpha)\lambda_{2,t}\tilde{P}_{H,t}^{-\eta-1}C_t\tilde{P}_t^{\eta} - \gamma\alpha\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma-1}C_F$$
$$+ \lambda_{3,t}(1-\alpha)(1-\eta)\tilde{P}_{H,t}^{-\eta} + \lambda_{4,t}Z_tN_t = 0, \tag{B.50}$$
$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = (1-\alpha)\eta\lambda_{2,t}\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta-1} - (1-\eta)\lambda_{3,t}\tilde{P}_t^{-\eta} - \lambda_{4,t}C_t = 0. \tag{B.51}$$

The first-order conditions (B.48)–(B.51), constraints (35)–(38), cooperation indicator $\mathbb{1}_{CP} = 0$, complete markets indicator $\mathbb{1}_{CM} = 0$, and exogenous shock dynamics (33) describe the full nonlinear dynamics of the system. To obtain an analytical expression, we must first consider the behavior of the model in the steady state.

B.3.2 Steady State

Solving for the optimal pricing constraint (35), the relationship between the terms of trade and the real exchange rate (37), goods and asset market clearing (36)–(38) in the steady state shows that $C = N = \tilde{P}_H = \tilde{P} = 1$. Substituting these values into the first-order conditions (B.48)–(B.51) yields the following steady-state relationships:

$$\frac{\partial \mathcal{L}}{\partial C} = 1 + \lambda_1(1-\sigma) + \lambda_2(1-\alpha) - \lambda_4 = 0, \tag{B.52}$$
$$\frac{\partial \mathcal{L}}{\partial N} = -1 - \lambda_1(1+\varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.53}$$
$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_H} = -\eta(1-\alpha)\lambda_2 - \gamma\alpha, \lambda_2 + \lambda_3(1-\alpha)(1-\eta) + \lambda_4 = 0, \tag{B.54}$$
$$\frac{\partial \mathcal{L}}{\partial \tilde{P}} = (1-\alpha)\eta\lambda_2 - (1-\eta)\lambda_3 - \lambda_4 = 0. \tag{B.55}$$

We solve the system (B.52)–(B.55) and obtain

$$\lambda_1 = -\frac{\alpha}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\eta\varphi - \alpha\eta\sigma}, \tag{B.56}$$
$$\lambda_2 = \frac{\varphi + \sigma}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\eta\varphi - \alpha\eta\sigma}, \tag{B.57}$$

$$\lambda_3 = \frac{\gamma(\varphi+\sigma)}{(\eta-1)(\alpha-\varphi-\sigma+\alpha\varphi+\eta\varphi+\gamma\varphi+\eta\sigma+\gamma\sigma-\alpha\eta\varphi-\alpha\eta\sigma)}, \tag{B.58}$$

$$\lambda_4 = \frac{(\varphi+\sigma)(\eta+\gamma-\alpha\eta)}{\alpha-\varphi-\sigma+\alpha\varphi+\eta\varphi+\gamma\varphi+\eta\sigma+\gamma\sigma-\alpha\eta\varphi-\alpha\eta\sigma}. \tag{B.59}$$

B.3.3 Log-Linearization

We log-linearize the first-order conditions (B.48)–(B.51) around the deterministic steady state and obtain

$$\begin{aligned} 0 &= -\sigma(1+\lambda_1(1-\sigma))\hat{C}_t + \lambda_2(1-\alpha)(\hat{\lambda}_{2,t} - \eta\hat{\tilde{P}}_{H,t} + \eta\hat{\tilde{P}}_t) \\ &\quad - \lambda_4(\hat{\lambda}_{4,t} + \hat{\tilde{P}}_t), \end{aligned} \tag{B.60}$$

$$0 = -(1+\lambda_1(1+\varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{\tilde{P}}_{H,t}), \tag{B.61}$$

$$\begin{aligned} 0 &= -\eta(1-\alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta+1)\hat{\tilde{P}}_{H,t} + \hat{C}_t + \eta\hat{\tilde{P}}_t) \\ &\quad - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma+1)\hat{\tilde{P}}_{H,t}) \\ &\quad + \lambda_3(1-\alpha)(1-\eta)(\hat{\lambda}_{3,t} - \eta\hat{\tilde{P}}_{H,t}) + \lambda_4(\hat{\lambda}_{4,t} + \hat{Z}_t + \hat{N}_t), \end{aligned} \tag{B.62}$$

$$\begin{aligned} 0 &= (1-\alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{\tilde{P}}_{H,t} + \hat{C}_t + (\eta-1)\hat{\tilde{P}}_t) \\ &\quad - (1-\eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{\tilde{P}}_t) - \lambda_4(\hat{\lambda}_{4,t} + \hat{C}_t). \end{aligned} \tag{B.63}$$

Now we log-linearize the constraints (35)–(38) after setting the value for the indicators $\mathbb{1}_{CP} = 0$, $\mathbb{1}_{CM} = 0$:

$$0 = -\hat{Z}_t - \hat{N}_t + (1-\alpha)(-\eta\hat{\tilde{P}}_{H,t} + \hat{C}_t + \eta\hat{\tilde{P}}_t) - \alpha\gamma\hat{\tilde{P}}_{H,t}, \tag{B.64}$$

$$0 = (1-\alpha)\hat{\tilde{P}}_{H,t} - \hat{\tilde{P}}_t, \tag{B.65}$$

$$0 = -\hat{C}_t - \hat{\tilde{P}}_t\hat{N}_t + \hat{Z}_t + \hat{\tilde{P}}_{H,t}. \tag{B.66}$$

We can express the system of linear equations consisting of (B.60)–(B.66) as

$$A(\theta)X_t + b(\theta)Z_t = 0, \tag{B.67}$$

where $X_t = [\hat{C}_t, \hat{N}_t, \hat{\tilde{P}}_{H,t}, \hat{\tilde{P}}_t, \hat{\lambda}_{2,t}, \hat{\lambda}_{3,t}, \hat{\lambda}_{4,t}]'$, A is a 6×6 matrix, and b is a 6×1 vector. After plugging in the values for $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, we can express the coefficients in A and b as functions of the model parameters $\theta = [\sigma, \varphi, \alpha, \eta, \gamma]$. As a result, we can express the endogenous variables X_t as a function of the parameter vector θ and the exogenous variable Z_t:

$$X_t = -A(\theta)^{-1} b(\theta) Z_t. \tag{B.68}$$

We can express the markup as a function of parameters θ and the technology shock Z_t by log-linearizing (25):

$$\mu_t = \hat{Z}_t + \hat{\tilde{P}}_{H,t} - \hat{\tilde{P}}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \tag{B.69}$$

B.3.4 Analytical Expression

Plugging the solution from (B.68) into (B.69) yields the following expression for the log-linear markup:

$$\hat{\mu}_t = \frac{G_3}{F_3} \hat{Z}_t, \tag{B.70}$$

where

$$G_3 = -\alpha(1-\alpha)(1+\varphi)(\eta-1)(\eta-1+\gamma), \tag{B.71}$$

$$\begin{aligned} F_3 = {} & 2\alpha\varphi - \sigma - \alpha\gamma - \varphi + \alpha\sigma + 3\eta\varphi + 3\gamma\varphi + 3\eta\sigma + 3\gamma\sigma \\ & + \alpha^2\eta - \alpha^3\eta + \alpha\gamma^2 - \alpha^2\varphi - 3\eta^2\varphi + \eta^3\varphi \\ & - 3\gamma^2\varphi + \gamma^3\varphi - 3\eta^2\sigma + \eta^3\sigma - 3\gamma^2\sigma + \gamma^3\sigma - \alpha^2\eta^2 \\ & + \alpha^3\eta^2 - 7\alpha^2\eta^2\varphi + 3\alpha^2\eta^3\varphi + 2\alpha^3\eta^2\varphi - \alpha^3\eta^3\varphi \\ & - 5\alpha^2\eta^2\sigma + 3\alpha^2\eta^3\sigma + \alpha^3\eta^2\sigma - \alpha^3\eta^3\sigma + \alpha\eta\gamma \\ & - 7\alpha\eta\varphi - 4\alpha\gamma\varphi - 5\alpha\eta\sigma - 2\alpha\gamma\sigma - 6\eta\gamma\varphi - 6\eta\gamma\sigma - \alpha^2\eta\gamma \\ & + 8\alpha\eta^2\varphi + 5\alpha^2\eta\varphi - 3\alpha\eta^3\varphi - \alpha^3\eta\varphi + 2\alpha\gamma^2\varphi + \alpha^2\gamma\varphi \\ & + 7\alpha\eta^2\sigma + 2\alpha^2\eta\sigma - 3\alpha\eta^3\sigma + \alpha\gamma^2\sigma + 3\eta\gamma^2\varphi + 3\eta^2\gamma\varphi \\ & + 3\eta\gamma^2\sigma + 3\eta^2\gamma\sigma + 8\alpha\eta\gamma\sigma - 3\alpha\eta\gamma^2\varphi - 6\alpha\eta^2\gamma\varphi - 4\alpha^2\eta\gamma\varphi \\ & - 3\alpha\eta\gamma^2\sigma - 6\alpha\eta^2\gamma\sigma - 2\alpha^2\eta\gamma\sigma + 3\alpha^2\eta^2\gamma\varphi \\ & + 3\alpha^2\eta^2\gamma\sigma + 10\alpha\eta\gamma\varphi. \end{aligned} \tag{B.72}$$

We log-linearize the export share (17) and use the production function (18) to obtain

$$\hat{E}_{s,t} = -\gamma\hat{\hat{P}}_{H,t} - \hat{Z}_t - \hat{N}_t. \tag{B.73}$$

Now substitute the solution for $\hat{N}_t$ and $\hat{\hat{P}}_{H,t}$ from (B.68) into (B.73) to get the log-linear export share:

$$\hat{E}_{s,t} = \frac{G_{3,E}}{F_{3,E}}\hat{Z}_t, \tag{B.74}$$

where

$$\begin{aligned}
G_{3,E} = {} & -(2\alpha + 3\eta + 2\gamma - \varphi - 7\alpha\eta - 3\alpha\gamma - 4\eta\gamma + 2\alpha\varphi + 3\eta\varphi \\
& + 2\gamma\varphi + 8\alpha\eta^2 + 5\alpha^2\eta - 3\alpha\eta^3 - \alpha^3\eta + \alpha\gamma^2 + \alpha^2\gamma + \eta\gamma^2 \\
& + 2\eta^2\gamma - \alpha^2\varphi - 3\eta^2\varphi + \eta^3\varphi - \gamma^2\varphi - \alpha^2 - 3\eta^2 + \eta^3 - \gamma^2 \\
& - 7\alpha^2\eta^2 + 3\alpha^2\eta^3 + 2\alpha^3\eta^2 - \alpha^3\eta^3 + 2\alpha^2\eta^2\gamma - 7\alpha^2\eta^2\varphi \\
& + 3\alpha^2\eta^3\varphi + 2\alpha^3\eta^2\varphi - \alpha^3\eta^3\varphi + 7\alpha\eta\gamma - 7\alpha\eta\varphi - 3\alpha\gamma\varphi \\
& - 4\eta\gamma\varphi - \alpha\eta\gamma^2 - 4\alpha\eta^2\gamma - 3\alpha^2\eta\gamma + 8\alpha\eta^2\varphi + 5\alpha^2\eta\varphi - 3\alpha\eta^3\varphi \\
& - \alpha^3\eta\varphi + \alpha\gamma^2\varphi + \alpha^2\gamma\varphi + \eta\gamma^2\varphi + 2\eta^2\gamma\varphi - \alpha\eta\gamma^2\varphi \\
& - 4\alpha\eta^2\gamma\varphi - 3\alpha^2\eta\gamma\varphi + 2\alpha^2\eta^2\gamma\varphi + 7\alpha\eta\gamma\varphi - 1), \qquad \text{(B.75)} \\
F_{3,E} = {} & 2\alpha\varphi - \sigma - \alpha\gamma - \varphi + \alpha\sigma + 3\eta\varphi + 3\gamma\varphi + 3\eta\sigma + 3\gamma\sigma + \alpha^2\eta \\
& - \alpha^3\eta + \alpha\gamma^2 - \alpha^2\varphi - 3\eta^2\varphi + \eta^3\varphi - 3\gamma^2\varphi + \gamma^3\varphi \\
& - 3\eta^2\sigma + \eta^3\sigma - 3\gamma^2\sigma + \gamma^3\sigma - \alpha^2\eta^2 + \alpha^3\eta^2 - 7\alpha^2\eta^2\varphi \\
& + 3\alpha^2\eta^3\varphi + 2\alpha^3\eta^2\varphi - \alpha^3\eta^3\varphi - 5\alpha^2\eta^2\sigma + 3\alpha^2\eta^3\sigma \\
& + \alpha^3\eta^2\sigma - \alpha^3\eta^3\sigma + \alpha\eta\gamma - 7\alpha\eta\varphi - 4\alpha\gamma\varphi - 5\alpha\eta\sigma - 2\alpha\gamma\sigma \\
& - 6\eta\gamma\varphi - 6\eta\gamma\sigma - \alpha^2\eta\gamma + 8\alpha\eta^2\varphi + 5\alpha^2\eta\varphi - 3\alpha\eta^3\varphi - \alpha^3\eta\varphi \\
& + 2\alpha\gamma^2\varphi + \alpha^2\gamma\varphi + 7\alpha\eta^2\sigma + 2\alpha^2\eta\sigma - 3\alpha\eta^3\sigma + \alpha\gamma^2\sigma \\
& + 3\eta\gamma^2\varphi + 3\eta^2\gamma\varphi + 3\eta\gamma^2\sigma + 3\eta^2\gamma\sigma + 8\alpha\eta\gamma\sigma - 3\alpha\eta\gamma^2\varphi \\
& - 6\alpha\eta^2\gamma\varphi - 4\alpha^2\eta\gamma\varphi - 3\alpha\eta\gamma^2\sigma - 6\alpha\eta^2\gamma\sigma - 2\alpha^2\eta\gamma\sigma \\
& + 3\alpha^2\eta^2\gamma\varphi + 3\alpha^2\eta^2\gamma\sigma + 10\alpha\eta\gamma\varphi. \qquad \text{(B.76)}
\end{aligned}$$

Then we can express the markup as a function of the export share using the formula

$$\frac{\hat{\mu}_t}{\hat{E}_{s,t}} = \frac{\frac{G_3}{F_3}}{\frac{G_{3,E}}{F_{3,E}}} = \frac{\alpha(\eta+\gamma-1)}{(\eta(1-\alpha)+\gamma-1)^2+\alpha(\eta(1-\alpha)+\gamma-1)}. \quad \text{(B.77)}$$

B.4 Proof of Proposition 4

B.4.1 Lagrangian

Under complete markets, noncooperative central banks maximize (34) subject to (35), (36), (37), and (38), where $\mathbb{1}_{CP} = 0$ and $\mathbb{1}_{CM} = 1$. The Lagrangian is

$$\begin{aligned}\mathcal{L} = \sum_{t=1}^{\infty} \beta^t \mathbb{E}\Bigg[&\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} + \lambda_1(E_{t-1}C_t^{1-\sigma} - E_{t-1}N_t^{1+\varphi}) \\ &+ \lambda_{2,t}((1-\alpha)\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta} + \alpha\tilde{P}_{H,t}^{-\gamma}C_F - Z_tN_t) \\ &+ \lambda_{3,t}((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha - \tilde{P}_t^{1-\eta}) \\ &+ \lambda_{4,t}\left(\mathbb{E}_{t-1}\{Z_tN_t\tilde{P}_{H,t}\} - C_t\tilde{P}_t^{\frac{1}{\sigma}}\mathbb{E}_{t-1}[\tilde{P}_t^{\frac{\sigma-1}{\sigma}}]\right)\Bigg]. \quad \text{(B.78)}\end{aligned}$$

The first-order conditions for the problem with respect to consumption C_t, labor N_t, the terms of trade $\tilde{P}_{H,t}$, and the real exchange rate $\tilde{P}_t$ are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial C_t} &= C_t^{-\sigma} + \lambda_1(1-\sigma)C_t^{-\sigma} + \lambda_{2,t}(1-\alpha)\tilde{P}_{H,t}^{-\eta}\tilde{P}_t^{\eta} \\ &\quad - \lambda_{4,t}\tilde{P}_t^{\frac{1}{\sigma}}\mathbb{E}[\tilde{P}_t^{\frac{\sigma-1}{\sigma}}], \quad \text{(B.79)}\\ \frac{\partial \mathcal{L}}{\partial N_t} &= -N_t^{\varphi} - \lambda_1(1+\varphi)N_t^{\varphi} - \lambda_{2,t}Z_t + \mathbb{E}(\lambda_{4,t})Z_t\tilde{P}_{H,t}, \quad \text{(B.80)}\\ \frac{\partial \mathcal{L}}{\partial \tilde{P}_{H,t}} &= -\eta(1-\alpha)\lambda_{2,t}\tilde{P}_{H,t}^{-\eta-1}C_t\tilde{P}_t^{\eta} - \gamma\alpha\lambda_{2,t}\tilde{P}_{H,t}^{-\gamma-1}C_F \\ &\quad + \lambda_{3,t}(1-\alpha)(1-\eta)\tilde{P}_{H,t}^{-\eta} + \mathbb{E}(\lambda_{4,t})Z_tN_t, \quad \text{(B.81)}\end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_t} = (1-\alpha)\eta\lambda_{2,t}\tilde{P}_{H,t}^{-\eta}C_t\tilde{P}_t^{\eta-1} - (1-\eta)\lambda_{3,t}\tilde{P}_t^{-\eta}$$
$$- \lambda_{4,t}\frac{1}{\sigma}\tilde{P}_t^{\frac{1}{\sigma}-1}C_t\mathbb{E}\tilde{P}_t^{1-\frac{1}{\sigma}} - \left(1-\frac{1}{\sigma}\right)\mathbb{E}\left[\lambda_{4,t}C_t\tilde{P}_t^{\frac{1}{\sigma}}\right]\tilde{P}_t^{-\frac{1}{\sigma}}. \tag{B.82}$$

These first-order conditions (B.79)–(B.82), constraints (35)–(38), cooperation indicator $\mathbb{1}_{CP} = 0$, complete markets indicator $\mathbb{1}_{CM} = 1$, and exogenous shock dynamics (33) describe the full nonlinear dynamics of the system. To obtain an analytical expression, we must first consider the behavior of the model in the steady state.

B.4.2 Steady State

Solving for the optimal pricing constraint (35), the relationship between the real exchange rate and the terms of trade (37), and goods and asset market clearing (36)–(38) in the steady state reveals that $C = N = \tilde{P}_H = \tilde{P} = 1$. Substituting these values into the first-order conditions (B.48)–(B.51) yields the following steady-state relationships:

$$\frac{\partial \mathcal{L}}{\partial C} = 1 + \lambda_1(1-\sigma) + \lambda_2(1-\alpha) - \lambda_4 = 0, \tag{B.83}$$

$$\frac{\partial \mathcal{L}}{\partial N} = -1 - \lambda_1(1+\varphi) - \lambda_2 + \lambda_4 = 0, \tag{B.84}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}_H} = -\eta(1-\alpha)\lambda_2 - \gamma\alpha\lambda_2 + \lambda_3(1-\alpha)(1-\eta) + \lambda_4, \tag{B.85}$$

$$\frac{\partial \mathcal{L}}{\partial \tilde{P}} = (1-\alpha)\eta\lambda_2 - (1-\eta)\lambda_3 - \lambda_4\frac{1}{\sigma} - \left(1-\frac{1}{\sigma}\right)\lambda_4. \tag{B.86}$$

We solve the system (B.83)–(B.86) and obtain

$$\lambda_1 = -\frac{\alpha}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\eta\varphi - \alpha\eta\sigma}, \tag{B.87}$$

$$\lambda_2 = \frac{\varphi + \sigma}{\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\eta\varphi - \alpha\eta\sigma}, \tag{B.88}$$

$$\lambda_3 = \frac{\gamma(\varphi + \sigma)}{(\eta - 1)(\alpha - \varphi - \sigma + \alpha\varphi + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - \alpha\eta\varphi - \alpha\eta\sigma)}, \tag{B.89}$$

$$\lambda_4 = \frac{(\varphi+\sigma)(\eta+\gamma-\alpha\eta)}{\alpha-\varphi-\sigma+\alpha\varphi+\eta\varphi+\gamma\varphi+\eta\sigma+\gamma\sigma-\alpha\eta\varphi-\alpha\eta\sigma}. \tag{B.90}$$

The steady state fully coincides with the steady state for cooperative policy under financial autarky.

B.4.3 Log-Linearization

We log-linearize the first-order conditions (B.48)–(B.51) around the deterministic steady state and obtain

$$\begin{aligned}
0 &= -\sigma(1+\lambda_1(1-\sigma))\hat{C}_t + \lambda_2(1-\alpha)(\hat{\lambda}_{2,t} - \eta\hat{\hat{P}}_{H,t} + \eta\hat{\hat{P}}_t)\\
&\quad - \lambda_4\left(\hat{\lambda}_{4,t} + \frac{1}{\sigma}\hat{\hat{P}}_t\right), && \text{(B.91)}\\
0 &= -(1+\lambda_1(1+\varphi))\varphi\hat{N}_t - \lambda_2(\hat{\lambda}_{2,t} + \hat{Z}_t) + \lambda_4(\hat{Z}_t + \hat{\hat{P}}_{H,t}), && \text{(B.92)}\\
0 &= -\eta(1-\alpha)\lambda_2(\hat{\lambda}_{2,t} - (\eta+1)\hat{\hat{P}}_{H,t} + \hat{C}_t + \eta\hat{\hat{P}}_t)\\
&\quad - \gamma\alpha\lambda_2(\hat{\lambda}_{2,t} - (\gamma+1)\hat{\hat{P}}_{H,t}) + \lambda_3(1-\alpha)(1-\eta)(\hat{\lambda}_{3,t} - \eta\hat{\hat{P}}_{H,t})\\
&\quad + \lambda_4(\hat{Z}_t + \hat{N}_t), && \text{(B.93)}\\
0 &= (1-\alpha)\eta\lambda_2(\hat{\lambda}_{2,t} - \eta\hat{\hat{P}}_{H,t} + \hat{C}_t + (\eta-1)\hat{\hat{P}}_t)\\
&\quad - (1-\eta)\lambda_3(\hat{\lambda}_{3,t} - \eta\hat{\hat{P}}_t) - \lambda_4\frac{1}{\sigma}\left(\hat{\lambda}_{4,t} + \left(\frac{1}{\sigma}-1\right)\hat{\hat{P}}_t + \hat{C}_t\right)\\
&\quad + \left(1-\frac{1}{\sigma}\right)\frac{1}{\sigma}\lambda_4\hat{\hat{P}}_t. && \text{(B.94)}
\end{aligned}$$

Now we log-linearize the constraints (35)–(38) after setting the value for the indicators $\mathbb{1}_{CP} = 0$, $\mathbb{1}_{CM} = 0$:

$$0 = -\hat{Z}_t - \hat{N}_t + (1-\alpha)(-\eta\hat{\hat{P}}_{H,t} + \hat{C}_t + \eta\hat{\hat{P}}_t) - \alpha\gamma\hat{\hat{P}}_{H,t}, \tag{B.95}$$

$$0 = (1-\alpha)\hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t, \tag{B.96}$$

$$0 = \hat{C}_t + \frac{1}{\sigma}\hat{\hat{P}}_t. \tag{B.97}$$

We can express the system of linear equations consisting of (B.91)–(B.97) as

$$A(\theta)X_t + b(\theta)Z_t = 0, \tag{B.98}$$

where $X_t = [\hat{C}_t, \hat{N}_t, \hat{\hat{P}}_{H,t}, \hat{\hat{P}}_t, \hat{\lambda}_{2,t}, \hat{\lambda}_{3,t}, \hat{\lambda}_{4,t}]'$, A is a 6×6 matrix, and b is a 6×1 vector. After plugging in the values for $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, we can express the coefficients in A and b as functions of the model parameters $\theta = [\sigma, \varphi, \alpha, \eta, \gamma]$. As a result, the endogenous variables X_t can be expressed as a function of the parameter vector θ and the exogenous variable Z_t:

$$X_t = -A(\theta)^{-1} b(\theta) Z_t. \tag{B.99}$$

We can express the markup as function of the parameters θ and the technology shock Z_t by log-linearizing (25):

$$\mu_t = \hat{Z}_t + \hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t - \varphi \hat{N}_t - \sigma \hat{C}_t. \tag{B.100}$$

B.4.4 Analytical Expression

Plugging the solution from (B.99) into (B.100) allows us to obtain the expression for markup:

$$\hat{\mu}_t = \frac{G_4}{F_4} \hat{Z}_t, \tag{B.101}$$

where

$$\begin{aligned}
G_4 = {} & \sigma\alpha(1-\alpha)(1+\varphi)((1-2\eta)(\eta\sigma-1)\alpha + (\eta-1)^2 \\
& + \eta\sigma(\gamma-1) + \eta^2(\sigma-1)), \\
F_4 = {} & 4\alpha\varphi - \sigma - \varphi + 3\alpha\sigma + \eta\varphi + \gamma\varphi + \eta\sigma + \gamma\sigma - 6\alpha^2\varphi \\
& + 4\alpha^3\varphi - \alpha^4\varphi - 3\alpha^2\sigma + \alpha^3\sigma + 2\alpha\eta^2\sigma^2 \\
& + 3\alpha^2\eta\sigma^2 - \alpha^3\eta\sigma^2 + \alpha\gamma^2\sigma^2 - 5\alpha\eta\varphi - 4\alpha\gamma\varphi - 5\alpha\eta\sigma \\
& - 2\alpha\gamma\sigma - 5\alpha^2\eta^2\sigma^2 + 3\alpha^3\eta^2\sigma^2 + 10\alpha^2\eta\varphi \\
& - 10\alpha^3\eta\varphi + 5\alpha^4\eta\varphi - \alpha^5\eta\varphi + 6\alpha^2\gamma\varphi - 4\alpha^3\gamma\varphi + \alpha^4\gamma\varphi \\
& - 2\alpha\eta\sigma^2 + 7\alpha^2\eta\sigma - 3\alpha^3\eta\sigma - \alpha\gamma\sigma^2 + \alpha^2\gamma\sigma
\end{aligned}$$

$$
\begin{aligned}
&- 2\alpha\eta\varphi\sigma - 2\alpha\gamma\varphi\sigma - \alpha^2\eta^2\varphi\sigma^2 + \alpha^2\eta^3\varphi\sigma^2 + 2\alpha^3\eta^2\varphi\sigma^2 \\
&- 3\alpha^3\eta^3\varphi\sigma^2 - \alpha^4\eta^2\varphi\sigma^2 + 3\alpha^4\eta^3\varphi\sigma^2 - \alpha^5\eta^3\varphi\sigma^2 \\
&- \alpha^2\gamma^2\varphi\sigma^2 + \alpha^2\gamma^3\varphi\sigma^2 + 3\alpha\eta\gamma\sigma^2 + 2\alpha\eta^2\varphi\sigma + 6\alpha^2\eta\varphi\sigma \\
&- 6\alpha^3\eta\varphi\sigma + 2\alpha^4\eta\varphi\sigma + 2\alpha\gamma^2\varphi\sigma + 4\alpha^2\gamma\varphi\sigma \\
&- 2\alpha^3\gamma\varphi\sigma - 3\alpha^2\eta\gamma\sigma^2 - 8\alpha^2\eta^2\varphi\sigma + 12\alpha^3\eta^2\varphi\sigma - 8\alpha^4\eta^2\varphi\sigma \\
&+ 2\alpha^5\eta^2\varphi\sigma - 4\alpha^2\gamma^2\varphi\sigma + 2\alpha^3\gamma^2\varphi\sigma \\
&- 2\alpha^2\eta\gamma\varphi\sigma^2 + 2\alpha^3\eta\gamma\varphi\sigma^2 + 4\alpha\eta\gamma\varphi\sigma + 3\alpha^2\eta\gamma^2\varphi\sigma^2 \\
&+ 3\alpha^2\eta^2\gamma\varphi\sigma^2 - 3\alpha^3\eta\gamma^2\varphi\sigma^2 - 6\alpha^3\eta^2\gamma\varphi\sigma^2 \\
&+ 3\alpha^4\eta^2\gamma\varphi\sigma^2 - 12\alpha^2\eta\gamma\varphi\sigma + 12\alpha^3\eta\gamma\varphi\sigma - 4\alpha^4\eta\gamma\varphi\sigma.
\end{aligned}
$$

We can log-linearize the export share (17) and use the production function (18) to obtain

$$\hat{E}_{s,t} = -\gamma\hat{\tilde{P}}_{H,t} - \hat{Z}_t - \hat{N}_t. \tag{B.102}$$

We substitute the solution for $\hat{N}_t$ and $\hat{\tilde{P}}_{H,t}$ from (B.99) into (B.102) to get the log-linear export share:

$$\hat{E}_{s,t} = \frac{G_{4,E}}{F_{4,E}}\hat{Z}_t. \tag{B.103}$$

We do not provide the full expressions for $G_{4,E}$ and $F_{4,E}$, as they are enormous and provide little value by themselves. Instead, we express the log-linear markup as a function of the log-linear export share using the formula

$$\frac{\hat{\mu}_t}{\hat{E}_{s,t}} = \frac{\frac{G_4}{F_4}}{\frac{G_{4,E}}{F_{4,E}}} = \frac{G_x}{F_x H_x}, \tag{B.104}$$

where

$$
\begin{aligned}
G_x &= \alpha\sigma(\sigma(1-2\alpha)\eta^2 + \gamma\sigma\eta - (1-\alpha)\eta(\sigma+2) + 1 - \alpha), && \text{(B.105)}\\
F_x &= \alpha - 1 + \sigma(\gamma - \alpha\eta), && \text{(B.106)}\\
H_x &= (\eta(1-\alpha) + \gamma - 1)(1-\alpha)^2 \\
&\quad + \alpha\sigma[\eta(\eta(1-\alpha) + 2(2\gamma-1))(1-\alpha) + \gamma(\gamma-1)]. && \text{(B.107)}
\end{aligned}
$$

Appendix C. Corollaries and Lemmas

C.1 Proof of Lemma 1

Log-linearization of equations (37), (36), (38), and (25) around the deterministic steady state, where we utilize the independence of the terms of trade and technology shocks across time, gives

$$\hat{Z}_t + \hat{N}_t = (1-\alpha)(-\eta\hat{P}_{H,t} + \hat{C}_t + \eta\hat{\hat{P}}_t) - \alpha\gamma\hat{P}_{H,t}, \tag{C.1}$$

$$\hat{\hat{P}}_t = (1-\alpha)\hat{\hat{P}}_{H,t}, \tag{C.2}$$

$$\hat{C}_t = -\mathbb{1}_{CM}\frac{1}{\sigma}\hat{\hat{P}}_t + (1-\mathbb{1}_{CM})(\hat{N}_t + \hat{Z}_t + \hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t), \tag{C.3}$$

$$\hat{\mu}_t = \hat{Z}_t + \hat{\hat{P}}_{H,t} - \hat{\hat{P}}_t - \sigma\hat{C}_t - \varphi\hat{N}_t. \tag{C.4}$$

Thus, for complete markets, we obtain the following dynamics for $\hat{Y}_t, \hat{C}_t, \hat{\hat{P}}_{H,t}, \hat{\hat{P}}_t$:

$$\hat{Y}_t = -\left[(1-\alpha)\left(\frac{1-\alpha}{\sigma} + \eta\alpha\right) + \alpha\gamma\right]\hat{\hat{P}}_{H,t}, \tag{C.5}$$

$$\hat{N}_t = -\left[(1-\alpha)\left(\frac{1-\alpha}{\sigma} + \eta\alpha\right) + \alpha\gamma\right]\hat{\hat{P}}_{H,t} - \hat{Z}_t, \tag{C.6}$$

$$\hat{C}_t = -\frac{1-\alpha}{\sigma}\hat{\hat{P}}_{H,t}, \tag{C.7}$$

$$\hat{\mu}_t = (1+\varphi)\hat{Z}_t + \left(1 + \varphi\left[(1-\alpha)\left(\frac{1-\alpha}{\sigma} + \eta\alpha\right) + \alpha\gamma\right]\right)\hat{\hat{P}}_{H,t}. \tag{C.8}$$

It is easy to see that $\frac{\partial\hat{Y}_t}{\partial\hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial\hat{N}_t}{\partial\hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial\hat{C}_t}{\partial\hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial\hat{\mu}_t}{\partial\hat{\hat{P}}_{H,t}} > 0$, and with respect to technology shocks we have $\frac{\partial\hat{Y}_t}{\partial\hat{Z}_t} = 0$, $\frac{\partial\hat{C}_t}{\partial\hat{Z}_t} = 0$, $\frac{\partial\hat{N}_t}{\partial\hat{Z}_t} < 0$, $\frac{\partial\hat{\mu}_t}{\partial\hat{Z}_t} > 0$. On the other hand, under financial autarky,

$$\hat{Y}_t = [(1-\alpha)(1-\eta) - \gamma]\hat{\hat{P}}_{H,t}, \tag{C.9}$$

$$\hat{N}_t = [(1-\alpha)(1-\eta) - \gamma]\hat{\hat{P}}_{H,t} - \hat{Z}_t, \tag{C.10}$$

$$\hat{C}_t = -[(1-\alpha)\eta + \gamma - 1]\hat{\hat{P}}_{H,t}, \tag{C.11}$$

$$\hat{\mu}_t = (1+\varphi)\hat{Z}_t + [\alpha(1-\sigma) + (\sigma+\varphi)(-(1-\alpha)(1-\eta)+\gamma)]\hat{P}_{H,t}. \tag{C.12}$$

In this case, $\frac{\partial \hat{Y}_t}{\partial \hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial \hat{N}_t}{\partial \hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial \hat{C}_t}{\partial \hat{\hat{P}}_{H,t}} < 0$, $\frac{\partial \hat{\mu}_t}{\partial \hat{\hat{P}}_{H,t}} > 0$, and with respect to technology shocks we have $\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} = 0$, $\frac{\partial \hat{C}_t}{\partial \hat{Z}_t} = 0$, $\frac{\partial \hat{N}_t}{\partial \hat{Z}_t} < 0$, $\frac{\partial \hat{\mu}_t}{\partial \hat{Z}_t} > 0$. ■

C.2 Proof of Lemma 2

Under flexible prices firms charge constant markups. Thus, for complete markets, we obtain the following dynamics for $\hat{Y}_t, \hat{C}_t, \hat{\hat{P}}_{H,t}, \hat{\hat{P}}_t$ by setting $\hat{\mu}_t = 0$ in the system (C.5)–(C.8):

$$\hat{Y}_t = \frac{\delta_z + \varphi\delta_z}{1+\varphi\delta_z}\hat{Z}_t, \tag{C.13}$$

$$\hat{N}_t = \frac{\delta_z - 1}{1+\varphi\delta_z}\hat{Z}_t, \tag{C.14}$$

$$\hat{C}_t = \frac{1-\alpha}{\sigma}\frac{1+\varphi}{1+\varphi\delta_z}\hat{Z}_t, \tag{C.15}$$

$$\hat{\hat{P}}_{H,t} = -\frac{1+\varphi}{1+\varphi\delta_z}\hat{Z}_t, \tag{C.16}$$

where $\delta_z = (1-\alpha)\left(\frac{1-\alpha}{\sigma} + \eta\alpha\right) + \alpha\gamma > 0$.

It is easy to see that $\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} > 0$, $\frac{\partial \hat{C}_t}{\partial \hat{Z}_t} > 0$, $\frac{\partial \hat{\hat{P}}_{H,t}}{\partial \hat{Z}_t} < 0$. On the other hand, under financial autarky,

$$\hat{Y}_t = \frac{(1+\varphi)\delta_{2,z}}{\alpha(1-\sigma)+(\sigma+\varphi)\delta_{2,z}}\hat{Z}_t, \tag{C.17}$$

$$\hat{N}_t = \frac{(1-\sigma)(\delta_{2,z}-\alpha)}{\alpha(1-\sigma)+(\sigma+\varphi)\delta_{2,z}}\hat{Z}_t, \tag{C.18}$$

$$\hat{C}_t = \frac{(1+\varphi)(\delta_{2,z}-\alpha)}{\alpha(1-\sigma)+(\sigma+\varphi)\delta_{2,z}}\hat{Z}_t, \tag{C.19}$$

$$\hat{P}_{H,t} = -\frac{1+\varphi}{\alpha(1-\sigma)+(\sigma+\varphi)\delta_{2,z}}\hat{Z}_t, \tag{C.20}$$

where $\delta_{2,z} = (1-\alpha)(\eta-1)+\gamma$. In this case, $\frac{\partial \hat{Y}_t}{\partial \hat{Z}_t} > 0$, $\frac{\partial \hat{C}_t}{\partial \hat{Z}_t} > 0$, $\frac{\partial \hat{\tilde{P}}_{H,t}}{\partial \hat{Z}_t} < 0$.

C.3 Proof of Lemma 4

In the steady state, noncooperative central banks maximize (34) subject to (36), (37), and (38), where $\mathbb{1}_{CP} = 0$ and $\mathbb{1}_{CM} = 0$ (asset market structure is irrelevant in the steady state). Thus, we can formulate a Lagrangian:

$$\frac{C^{1-\sigma}}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi} + \lambda_2((1-\alpha)T^{-\eta}CP^{\eta} + \alpha T^{-\gamma}C_F - N) \\ + \lambda_3((1-\alpha)T^{1-\eta} + \alpha - P_t^{1-\eta}) + \lambda_4(NT - CP). \tag{C.21}$$

The first-order conditions for the problem with respect to consumption C_t, labor N_t, terms of trade $\tilde{P}_{H,t}$, and real exchange rate $\tilde{P}_t$ are given below:

$$\frac{\partial L}{\partial C} = C^{-\sigma} + \lambda_2(1-\alpha)T^{-\eta}P^{\eta} - \lambda_4 P, \tag{C.22}$$

$$\frac{\partial L}{\partial N} = -N^{\varphi} - \lambda_2 + \lambda_4 T, \tag{C.23}$$

$$\frac{\partial L}{\partial T} = -\eta(1-\alpha)\lambda_2 T^{-\eta-1}CP^{\eta} - \gamma\alpha\lambda_2 T^{-\gamma-1}C_F \\ + \lambda_3(1-\alpha)(1-\eta)T^{-\eta} + \lambda_4 N, \tag{C.24}$$

$$\frac{\partial L}{\partial P} = (1-\alpha)\eta\lambda_2 T^{-\eta}CP^{\eta-1} - (1-\eta)\lambda_3 P^{-\eta} - \lambda_4 C = 0. \tag{C.25}$$

C.3.1 Symmetric Steady State

In the symmetric steady state we use the fact that $T = P = Z = 1$ and $C = N$, then use these relationships to simplify first-order conditions and obtain

$$\frac{\partial L}{\partial C_t} = C^{-\sigma} + \lambda_2(1-\alpha) - \lambda_4 = 0, \tag{C.26}$$

$$\frac{\partial L}{\partial N_t} = -C^{\varphi} - \lambda_2 + \lambda_4 = 0, \tag{C.27}$$

$$\frac{\partial L_t}{\partial T_t} = -\eta(1-\alpha)\lambda_2 C - \gamma\alpha\lambda_2 C + \lambda_3(1-\alpha)(1-\eta) + \lambda_4 C, \quad (C.28)$$

$$\frac{\partial L}{\partial P_t} = (1-\alpha)\eta\lambda_2 C - (1-\eta)\lambda_3 - \lambda_4 C. \quad (C.29)$$

We simplify this system and use $\frac{\partial L}{\partial P_t}(1-\alpha) + \frac{\partial L_t}{\partial T_t}$:

$$-\alpha(\eta(1-\alpha)+\gamma)\lambda_2 + \alpha\lambda_4 = 0. \quad (C.30)$$

We can express λ_4 as a function from λ_2 from (C.30) and obtain $\lambda_4 = (\eta(1-\alpha)+\gamma)\lambda_2$. We substitute this result into the first-order conditions (C.26)–(C.27) with respect to C and N:

$$C^{-\sigma} = ((\eta-1)(1-\alpha)+\gamma)\lambda_2, \quad (C.31)$$

$$C^{\varphi} = (\eta(1-\alpha)+\gamma-1)\lambda_2. \quad (C.32)$$

Expressing λ_2 in terms of C, γ, η, α, and making these expressions equal allows us to obtain

$$C^{-\sigma}\frac{1}{((\eta-1)(1-\alpha)+\gamma)} = \frac{C^{\varphi}}{(\eta(1-\alpha)+\gamma-1)}. \quad (C.33)$$

Thus, we can express C in terms of α, η, γ:

$$C^{-\sigma-\varphi} = \frac{(\eta-1)(1-\alpha)+\gamma}{\eta(1-\alpha)+\gamma-1} = 1 + \frac{\alpha}{\eta(1-\alpha)+\gamma-1}. \quad (C.34)$$

As the relationship (25) in the steady state takes the form

$$C^{-\sigma-\varphi} = \mu, \quad (C.35)$$

we can obtain the formula for the markup:

$$\mu = 1 + \frac{\alpha}{\eta(1-\alpha)+\gamma-1}. \quad (C.36)$$

C.4 Proof of Corollary 4.2

We begin with the expression for markup:

$$\hat{\mu}_t = \frac{G_x}{F_x H_x}\hat{E}_{s,t}, \quad (C.37)$$

where

$$G_x = \alpha\sigma(\sigma(1-2\alpha)\eta^2 + \gamma\sigma\eta - (1-\alpha)\eta(\sigma+2) + 1 - \alpha), \quad \text{(C.38)}$$

$$F_x = \alpha - 1 + \sigma(\gamma - \alpha\eta), \quad \text{(C.39)}$$

$$\begin{aligned} H_x &= (\eta(1-\alpha) + \gamma - 1)(1-\alpha)^2 \\ &\quad + \alpha\sigma[\eta(\eta(1-\alpha) + 2(2\gamma-1))(1-\alpha) + \gamma(\gamma-1)]. \end{aligned} \quad \text{(C.40)}$$

We start with the fact that for $\eta > 0, \gamma \geq 1, 0 < \alpha < 1$, we have $H_x > 0$. Second, using the solution from (B.99) and (B.103) we can express consumption dynamics in terms of export share and obtain

$$\hat{E}_{s,t} = \frac{(1-\alpha)(\alpha - 1 + \sigma(\gamma - \alpha\eta))}{(\alpha-1)^2 + \alpha\sigma(\eta(1-\alpha) + \gamma)}\hat{Y}_t. \quad \text{(C.41)}$$

The export share is procyclical when $\alpha - 1 + \sigma(\gamma - \alpha\eta) > 0$, which is equivalent to $F_x > 0$.

Let's rearrange the expression $\frac{G_x}{F_x}$ and consider its simplified form:

$$\frac{G_x}{F_x} = \eta + \frac{(1-\alpha)(\eta - \frac{1}{\sigma})(\eta - 1)}{(1-\alpha)(\eta - \frac{1}{\sigma}) + (\gamma - \eta)}. \quad \text{(C.42)}$$

As $F_x > 0$, the denominator in equation (C.42) is positive. If $\sigma > 1, \gamma \geq 1, 0 < \alpha < 1$, we have to consider several cases for η. For example, if $\eta > \gamma$ and $F_x > 0$, then $(1-\alpha)(\eta - \frac{1}{\sigma})(\eta - 1) > 0$ and $\frac{G_x}{F_x} > 0$. Second, if $1 < \eta < \gamma$, then $\frac{G_x}{F_x}$ since the denominator and the numerator in equation (C.42) are positive. Third, it might be the case that $\frac{1}{\sigma} < \eta < 1$. In this case, the denominator in (C.42) is positive, while the numerator is negative. The whole expression $\frac{G_x}{F_x}$ monotonically increases with α. Let's take a derivative with respect to α:

$$\begin{aligned} &\frac{\begin{array}{c} -(\eta - \frac{1}{\sigma})(\eta - 1)[(1-\alpha)(\eta - \frac{1}{\sigma}) + \gamma - \eta) \\ +(\eta - \frac{1}{\sigma})(1-\alpha)(\eta - \frac{1}{\sigma})(\eta - 1) \end{array}}{F_x^2} \\ &= \frac{-(\eta - 1/\sigma)(\eta - 1)(\gamma - \eta)}{F_x^2} > 0. \end{aligned} \quad \text{(C.43)}$$

Then let's consider what happens at $\alpha = 0$, which sets $\frac{F_x}{G_x}$ to a minimum. In this case, we have

$$\frac{F_x}{G_x} = \eta + \frac{(\eta - \frac{1}{\sigma})(\eta - 1)}{\gamma - \frac{1}{\sigma}}. \tag{C.44}$$

We can multiply everything by the positive denominator and obtain the following expression:

$$\eta\left(\gamma - \frac{1}{\sigma}\right) + \eta^2 - \left(1 + \frac{1}{\sigma}\right)\eta + \frac{1}{\sigma}, \tag{C.45}$$

which is equivalent to

$$\left(\eta - \frac{1}{\sigma}\right)^2 + \eta(\gamma - 1) > 0. \tag{C.46}$$

Now we also technically have the case when $\eta < 1/\sigma$. In this case, as $F_x > 0$ and the numerator in (C.42) is positive, we obtain $\frac{G_x}{F_x} > 0$. Thus, markups are procyclical if export share is procyclical for households with risk aversion greater than one.

C.5 Proof of Corollary 4.3

We begin with the expression for markup:

$$\hat{\mu}_t = \frac{G_x}{F_x H_x}\hat{E}_{s,t}, \tag{C.47}$$

where

$$G_x = \alpha\sigma(\sigma(1 - 2\alpha)\eta^2 + \gamma\sigma\eta - (1 - \alpha)\eta(\sigma + 2) + 1 - \alpha), \tag{C.48}$$

$$F_x = \alpha - 1 + \sigma(\gamma - \alpha\eta), \tag{C.49}$$

$$\begin{aligned} H_x &= (\eta(1 - \alpha) + \gamma - 1)(1 - \alpha)^2 \\ &\quad + \alpha\sigma[\eta(\eta(1 - \alpha) + 2(2\gamma - 1))(1 - \alpha) + \gamma(\gamma - 1)]. \end{aligned} \tag{C.50}$$

As $H_x > 0$, we need to focus on G_x and F_x. Under $\eta = \gamma$, we can transform the expression to $\hat{\mu}_t$ to

$$\frac{\hat{\mu}_t}{\hat{E}_{s,t}} = \frac{\alpha\sigma(1 - \alpha)(2\gamma - 1)(\sigma\gamma - 1)}{(1 - \alpha)(\sigma\gamma - 1)H_x} = \frac{\alpha\sigma(2\gamma - 1)}{H_x} > 0. \tag{C.51}$$

Appendix D. Monetary Policy and Terms of Trade

The Euler equation for domestic home securities gives the following relationship:

$$C_t^{-\sigma} = \beta R_t E_t \frac{C_{t+1}^{-\sigma}}{\pi_{t+1}}. \tag{D.1}$$

We can express inflation in the consumer price index in terms of producer price index in the following way:

$$\pi_{t+1} = \frac{P_{F,t+1}}{P_{F,t}} \frac{\tilde{P}_{t+1}}{\tilde{P}_t} = \pi_{H,t+1} \frac{\tilde{P}_{t+1} \tilde{P}_{H,t}}{\tilde{P}_t \tilde{P}_{H,t+1}}. \tag{D.2}$$

Note that inflation $\pi_{H,t+1}$ is predetermined at t. Plugging this inflation in the original Euler equation gives

$$\frac{\tilde{P}_{H,t}}{\tilde{P}_t} C_t^{-\sigma} = \beta \frac{R_t}{\pi_{H,t+1}} E_t \left[C_{t+1}^{-\sigma} \frac{\tilde{P}_{H,t+1}}{\tilde{P}_{t+1}} \right]. \tag{D.3}$$

Under complete markets we can rely on the expression for consumption

$$C_{it} = \frac{\mathbb{E}\left\{ Y_{it} \tilde{P}_{H,it} \right\}}{\mathbb{E}\left\{ \tilde{P}_{it}^{\frac{\sigma-1}{\sigma}} \right\}} \tilde{P}_{it}^{-\frac{1}{\sigma}} \tag{D.4}$$

and obtain

$$\tilde{P}_{H,t} = \beta \frac{R_t}{\pi_{H,t+1}} \mathbb{E}_t \tilde{P}_{H,t+1}. \tag{D.5}$$

When the prices are set one period in advance, inflation dynamics on its own has no effect on the real economy. As a result, we are not interested in pinning down the inflation dynamics, using the dynamics of the real interest rate $r_t = \frac{R_t}{\pi_{H,t+1}}$ for finding out the equilibrium of the real variables. Therefore, our objective is to generate a real interest rate rule that generates the path for the terms of trade $\tilde{P}_{H,t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} E\tilde{P}_{H,t}$. This is not a trivial exercise, as we cannot rely on linear approximations due to the fact that we claim later in the paper to arrive at the optimality conditions nonlinearly.

Moreover, our interest rate rule should rule out multiple equilibriums. The following rule produces a unique stationary equilibrium that generates the necessary terms-of-trade dynamics:

$$r_t = \frac{1}{\beta} \frac{\tilde{P}_{H,t}^{-\delta}(\frac{g(Z_t)}{\mathbb{E}g(Z_t)})^{1+\delta}}{\mathbb{E}[\tilde{P}_{H,t}^{-\delta}(\frac{g(Z_t)}{\mathbb{E}g(Z_t)})^{1+\delta}]}, \tag{D.6}$$

where δ is a positive constant. We use this equation, as the more simple rule $r_t = \frac{g(Z_t)}{\mathbb{E}g(Z_t)}$ cannot rule out sunspot terms for the terms of trade. This rule for the interest rate allows us to obtain the modified relationship for the terms of trade:

$$\tilde{P}_{H,t} = \frac{\frac{g(Z_t)}{\mathbb{E}g(Z_t)}}{\mathbb{E}[\tilde{P}_{H,t}^{-\delta}(\frac{g(Z_t)}{\mathbb{E}g(Z_t)})^{1+\delta}]^{\frac{1}{1+\delta}}} \left(E_t \tilde{P}_{H,t+1}\right)^{\frac{1}{1+\delta}}. \tag{D.7}$$

For this equation, the only non-explosive solution for the terms-of-trade dynamics is $\tilde{P}_{H,t} = \frac{g(Z_t)}{\mathbb{E}(g(Z_t))} E\tilde{P}_{H,t}$.

Under financial autarky, we return to the original Euler equation and set the interest rate rule as $R_t = \frac{\pi_{H,t+1}}{\beta} \frac{g(Z_t)}{\mathbb{E}g(Z_t)}$; then, using the independence of shocks across time generates

$$C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{\tilde{P}_t} = \frac{g(Z_t)}{\mathbb{E}g(Z_t)} E\left[C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{\tilde{P}_t}\right]. \tag{D.8}$$

Having the control over the cyclical dynamics $C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{\tilde{P}_t}$, coupled with the constraints (36)–(38) allows us to extract the dynamics for the terms of trade. For example, we can express consumption under financial autarky using (36)–(38) as

$$C_t = ((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha)^{\frac{-\eta}{1-\eta}} \tilde{P}_{H,t}^{1-\gamma} A, \tag{D.9}$$

where A is a constant. Using this relationship, we can obtain the expression for $C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{\tilde{P}_t}$:

$$C_t^{-\sigma} \frac{\tilde{P}_{H,t}}{\tilde{P}_t} = ((1-\alpha)\tilde{P}_{H,t}^{1-\eta} + \alpha)^{\frac{\sigma\eta-1}{1-\eta}} \tilde{P}_{H,t}^{\gamma\sigma-\sigma+1} A^{-\sigma}. \tag{D.10}$$

Thus, the desire to replicate a particular cyclical dynamics of the terms of trade $\frac{\tilde{P}_{H,t}}{\mathbb{E}\tilde{P}_{H,t}} = \frac{f(Z_t)}{\mathbb{E}f(Z_t)}$ requires us to generate the dynamics for

$$\frac{C_t^{-\sigma}\frac{\tilde{P}_{H,t}}{\tilde{P}_t}}{\mathbb{E}[C_t^{-\sigma}\frac{\tilde{P}_{H,t}}{\tilde{P}_t}]} = \frac{((1-\alpha)f(Z_t)^{1-\eta} + \alpha\mathbb{E}f(Z_t))^{\frac{\sigma\eta-1}{1-\eta}}f(Z_t)^{\gamma\sigma-\sigma+1}}{\mathbb{E}\left[((1-\alpha)f(Z_t)^{1-\eta} + \alpha\mathbb{E}f(Z_t))^{\frac{\sigma\eta-1}{1-\eta}}f(Z_t)^{\gamma\sigma-\sigma+1}\right]}. \tag{D.11}$$

Defining $g(Z_t)$ as $((1-\alpha)f(Z_t)^{1-\eta} + \alpha Ef(Z_t))^{\frac{\sigma\eta-1}{1-\eta}}f(Z_t)^{\gamma\sigma-\sigma+1}$, we can use a similar rule,

$$r_t = \frac{1}{\beta}\frac{(C_t^{-\sigma}\frac{\tilde{P}_{H,t}}{\tilde{P}_t})^{-\delta}(\frac{g(Z_t)}{\mathbb{E}g(Z_t)})^{1+\delta}}{\mathbb{E}[(C_t^{-\sigma}\frac{\tilde{P}_{H,t}}{\tilde{P}_t})^{-\delta}(\frac{g(Z_t)}{\mathbb{E}g(Z_t)})^{1+\delta}]}. \tag{D.12}$$

Consequently, the central bank can control the terms of trade over the cycle under financial autarky, and when markets are complete.

Reference

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