Online Appendices to “Optimal Inflation Rates in a Non-linear New Keynesian Model: The Case of Japan and the United States”

Tomohide Mineyama,[a] Wataru Hirata,[b] and Kenji Nishizaki[b]
[a]International Monetary Fund
[b]Bank of Japan

Appendix A. Allocation in Cashless Economy under Flexible Prices and Wages

Under flexible prices and wages, the labor market equilibrium determines the relationship between the marginal product of labor and the marginal rate of substitution of labor for consumption:

$$\left(\frac{\theta}{\theta-1}\right)^{-1} A_t Z_t = \chi_t \frac{\left(H_t^f\right)^{1/\eta}}{\Xi_t^f}, \tag{A.1}$$

where the variables with a superscript f denote the endogenous variables in the cashless economy under flexible prices and wages. The right-hand side of (A.1) is the marginal rate of substitution whereas $A_t Z_t$ in the left-hand side is the marginal product of labor. Note that the allocation is still inefficient due to monopolistic distortion of firms represented by the steady-state markup $\theta/(\theta-1)$. To consider the cashless economy, we disregard the transaction cost of purchasing goods. Along with the market clearing conditions, (A.1) yields

$$Y_t^f = A_t Z_t \left(\frac{\theta}{\theta-1}\chi_t\right)^{-\frac{1}{1+1/\eta}}, \tag{A.2}$$

$$C_t^f = A_t Z_t \left(\frac{\theta}{\theta-1}\chi_t\right)^{-\frac{1}{1+1/\eta}}, \tag{A.3}$$

$$H_t^f = \left(\frac{\theta}{\theta-1}\chi_t\right)^{-\frac{1}{1+1/\eta}}. \tag{A.4}$$

Notice that risk premium Q_t does not affect Y_t^f because the allocation is determined only by intratemporal equilibrium conditions.

Appendix B. Model Solution

We employ a version of the policy function iteration method of Coleman (1990) to solve our non-linear model.[1] Specifically, we use the fixed-point iteration method. Richter, Throckmorton, and Walker (2014) found that this method has an advantage in terms of the speed of computation compared with alternative methods.

The concept of the fixed-point iteration method is summarized as follows. A model has the representation:

$$0 = \mathbb{E}_t \left[f \left(\mathbf{S}_t, \mathbf{X}_t, \mathbf{S}_{t+1}, \mathbf{X}_{t+1} \right) \right],$$

where $\mathbf{X}_t$ is a set of jump variables and $\mathbf{S}_t$ is a set of state variables. The intertemporal and intratemporal relationships among variables are represented in $f(\cdot)$. $\mathbb{E}_t[\cdot]$ is the expectation operator conditional on the information available at time t. In our baseline model, $\mathbf{X}_t = \left\{ Y_t, H_t, C_t, S_t, M_t, T_t, Y_t^f, P_t \right\}$ and $\mathbf{S}_t = \{D_{t-1}, W_{t-1}, R_{n,t-1}, A_t, Z_t, \chi_t, Q_t\}$. Jump variables and future state variables can be expressed as function of current state variables in the rational expectation equilibrium. Therefore, the model equations above can be rewritten as below:

$$\begin{aligned} 0 &= \mathbb{E}_t \left[f \left(\mathbf{S}_t, \left(\mathbf{S}_t \right), S\left(\mathbf{S}_t \right), \left(S \left(\mathbf{S}_t \right) \right) \right) \right] \\ &= \mathbb{E}_t \left[f \left(\Phi \left(\mathbf{S}_t \right) \right) \right], \end{aligned}$$

where $X(\cdot)$ and $S(\cdot)$ are the time-invariant policy functions for jump and state variables, which are summarized in $\Phi(\cdot)$. The model conditions $f(\cdot)$ and the policy function $\Phi(\cdot)$ are non-linear in general. The policy function iteration method discretizes the state space for $\mathbf{S}_t$ and numerically searches for the mapping $\Phi(\mathbf{S}_t)$ that satisfies the model equations. Consequently, the method is robust to the non-linearity of the underlying function $f(\cdot)$.

Algorithm. The algorithm takes the following steps in each iteration $n = 1, 2, 3...$

1. Formulate the initial guess for the policy functions $\Phi^{(0)}(\mathbf{S}_t)$.

[1]Similar methods are used by Katagiri (2016) and Iiboshi, Shintani, and Ueda (forthcoming) to solve a New Keynesian model with ZLB.

2. Substitute the previous guess $\Phi^{(n-1)}(\mathbf{S}_t)$ into the model equations to obtain the updated policy function $\Phi^{(n)}(\mathbf{S}_t)$. The parameter for updating $\alpha \in (0,1)$ is set equal to 0.2.[2] In this step, we approximate the future variables by using the linear interpolation method between grids and evaluate the expectation operator by numerical integration:

$$\Phi^*(\mathbf{S}_t) = \mathbb{E}_t\left[f\left(\Phi^{(n-1)}(\mathbf{S}_t)\right)\right] + \Phi^{(n-1)}(\mathbf{S}_t),$$
$$\Phi^{(n)}(\mathbf{S}_t) = \alpha\Phi^*(\mathbf{S}_t) + (1-\alpha)\,\Phi^{(n-1)}(\mathbf{S}_t).$$

3. Compute the deviations between the updated and previous policy functions:

$$dist = \max\left|\Phi^{(n)}(\mathbf{S}_t) - \Phi^{(n-1)}(\mathbf{S}_t)\right|.$$

4. Stop iterations if the deviation $dist$ becomes smaller than the critical value $\epsilon > 0$. Otherwise, go back to Step 2. We set $\epsilon = 10^{-4}$.

Appendix C. Deterministic and Risky Steady States

Recent studies argued that in a non-linear and stochastic environment the deterministic steady state (DSS) does not necessarily accord with the risky steady state (RSS) since agents take precautionary behavior even when the risk is not materialized (e.g., Coeurdacier, Rey, and Winant 2011). For example, Hills, Nakata, and Schmidt (2019) found that the inflation rate in the RSS is lower than in the DSS in the presence of ZLB risk by as much as 0.5 percent in annual rate in a New Keynesian model.

In our model, the regime-switching component of risk premium Q_t^{rs} takes either high or low values, and therefore policy function

[2] Richter, Throckmorton, and Walker (2014) pointed out that a smaller value of α helps maintain stability of solution, especially at the beginning of the algorithm, whereas it involves a larger number of iterations until convergence. Therefore, they proposed to use a small value of α for solving a large model. We set the value of α reflecting the trade-off between speed and stability of computation. However, it should be noted that the convergence criteria of policy function is independent of the value of α.

is not defined on $Q_t^{rs} = 1$. Hence, we define "quasi-RSS" as the unconditional mean of the RSS in each regime.[3] Given the conceptual difference in the definition of the RSS from the one analyzed by previous studies, the results here should be taken with some reservation.

Table C.1 reports the values of key variables in the DSS and quasi-RSS in our model. The quasi-RSS is computed for the specifications in which each element is sequentially added into the model. When only the nominal price rigidity is present in the model, it is notable that the inflation rate in the quasi-RSS is non-trivially higher than that in the DSS. The non-linearity in the Phillips curve implies that firms' price setting is convex in terms of aggregate inflation. Therefore, the presence of uncertainty leads to higher expected inflation in the future, rendering the current inflation rate higher as well. The effect is more apparent in the case of the United States where the calibrated Π^* is higher. Adding DNWR further raises the inflation rate in the quasi-RSS because it prevents the declines in the inflation rate upon contractionary shocks. On the other hand, adding the ZLB lowers the inflation rate in the quasi-RSS as Hills, Nakata, and Schmidt (2019) demonstrated. Regarding real quantities, consumption and labor input in the quasi-RSS become substantially lower when each element is added in the model because agents increase precautionary saving in the presence of these frictions.

[3]Formally, we define the quasi-RSS X^{rss} as below:

$$
\begin{aligned}
& X^{rss} = p_1 X_1^{rss} + p_2 X_2^{rss}, \\
\text{where}\quad & \Psi_1^{rss} = f(\Psi_1^{rss}, \Theta^{ss}, Q_t^{rs} = Q_1^{rs}), \\
& \Psi_2^{rss} = f(\Psi_2^{rss}, \Theta^{ss}, Q_t^{rs} = Q_1^{rs}).
\end{aligned}
$$

p_i is the unconditional probability of regime i for $i = 1, 2$. $f(\cdot)$ is the policy function. Ψ and Θ collect endogenous and exogenous variables, respectively. On the other hand, the DSS is defined as the allocation achieved when the regime-switching component is muted, i.e., $Q^{rs} = 1$.

Table C.1. Deterministic and Risky Steady States

Variable → **Symbol →**	**Consumption** C	**Labor Input** L	**Price Dispersion** V	**Inflation Rate** π
A. Japan				
Deterministic S.S.	0.93	0.93	1.00	0.38
Quasi-Risky S.S. (Diff. from Deterministic S.S.)				
Price Rigidity Only	0.93 (–0.08)	0.93 (+0.00)	1.00 (+0.08)	0.60 (+0.22)
Add Money Holdings	0.93 (–0.14)	0.93 (–0.06)	1.00 (+0.08)	0.64 (+0.26)
Add DNWR	0.92 (–1.12)	0.92 (–1.07)	1.00 (+0.05)	0.99 (+0.61)
Add ZLB	0.91 (–2.79)	0.91 (–2.75)	1.00 (+0.04)	0.29 (–0.09)
RW Rule	0.93 (–0.62)	0.93 (–0.61)	1.00 (+0.01)	0.64 (+0.26)
B. United States				
Deterministic S.S.	0.93	0.93	1.00	0.38
Quasi-Risky S.S. (Diff. from Deterministic S.S.)				
Price Rigidity Only	0.93 (–0.08)	0.93 (+0.00)	1.00 (+0.08)	0.60 (+0.22)
Add Money Holdings	0.93 (–0.14)	0.93 (–0.06)	1.00 (+0.08)	0.64 (+0.26)
Add DNWR	0.92 (–1.12)	0.92 (–1.07)	1.00 (+0.05)	0.99 (+0.61)
Add ZLB	0.91 (–2.79)	0.91 (–2.75)	1.00 (+0.04)	0.29 (–0.09)
RW Rule	0.93 (–0.62)	0.93 (–0.61)	1.00 (+0.01)	0.64 (+0.26)

Note: The inflation rate and the nominal interest rate are expressed in annual rates, whereas the other variables are in levels. The inflation rate in the deterministic steady state is calibrated according to the mean inflation rate during the sample period of 1985:Q1–2017:Q4 for Japan and 1987:Q4–2017:Q4 for the United States. The deterministic steady state takes money holdings into account. The allocation is not affected by DNWR or the ZLB.

Appendix D. Linearized Equilibrium Conditions

Deterministic Steady State. The allocation in the deterministic steady state is described below.

$$R^* = \frac{g}{\beta}, \tag{D.1}$$

$$R_n^* = \frac{g\Pi^*}{\beta}, \tag{D.2}$$

$$V^* = \sqrt{\frac{\delta_2 + 1 - 1/R_n^*}{\delta_1}}, \tag{D.3}$$

$$\left(\frac{M}{P}\right)^* = \frac{C^*}{V^*}, \tag{D.4}$$

$$\widetilde{\Xi}^* = \frac{1}{\widetilde{C}^* \left(1 + s(V^*) + V^* s'(V^*)\right)}, \tag{D.5}$$

$$\left(\frac{B}{P}\right)^* = \left(\frac{1-\lambda}{1-\lambda(\Pi^*)^{\theta-1}}\right)^{\frac{1}{\theta-1}}, \tag{D.6}$$

$$\left(\frac{\widetilde{W}}{P}\right)^* = MC^* = \frac{\theta-1}{\theta}\left(\frac{1-\lambda\beta(\Pi^*)^{\theta}}{1-\lambda\beta(\Pi^*)^{\theta-1}}\right)\left(\frac{1-\lambda}{1-\lambda(\Pi^*)^{\theta-1}}\right)^{\frac{1}{\theta-1}}, \tag{D.7}$$

$$\Omega_1^* = \frac{\theta}{\theta-1}\frac{MC^*\widetilde{\Xi}^*\widetilde{Y}^*}{1-\lambda\beta(\Pi^*)^{\theta}}, \tag{D.8}$$

$$\Omega_2^* = \frac{\widetilde{\Xi}^*\widetilde{Y}^*}{1-\lambda\beta(\Pi^*)^{\theta-1}}, \tag{D.9}$$

$$D^* = \left(\frac{1-\lambda}{1-\lambda(\Pi^*)^{\theta}}\right)\left(\frac{1-\lambda(\Pi^*)^{\theta-1}}{1-\lambda}\right)^{\frac{\theta}{\theta-1}}, \tag{D.10}$$

$$\widetilde{Y}^* = \left\{MC^*(D^*)^{1/\eta}\left(1 + s(V^*) + V^* s'(V^*)\right)\left(1 + s(V^*)\right)\right\}^{\frac{1}{1+1/\eta}}, \tag{D.11}$$

$$H^* = \widetilde{Y}^* D^*, \tag{D.12}$$

$$\widetilde{C}^* = \frac{\widetilde{Y}^*}{1+s(V^*)}, \tag{D.13}$$

$$\widetilde{Y}^{f*} = \left(\frac{\theta}{\theta-1}\right)^{-\frac{1}{1+1/\eta}}, \tag{D.14}$$

where X^* are the value in the deterministic steady state. Variables with tildes are detrended by real trend growth, i.e., $\widetilde{X}_t = X_t/A_t$. Exogenous variables Z_t, χ_t, and Q_t are normalized at 1 in the steady state.

Piecewise Log-Linearized Equilibrium Conditions. Piecewise log-linearized equilibrium conditions are listed below.

$$\widehat{\widetilde{\Xi}}_t = \mathbb{E}_t\left[\widehat{\widetilde{\Xi}}_{t+1}\right] + \widehat{R}_t - \mathbb{E}_t\left[\widehat{\Pi}_{t+1}\right] + \widehat{Q}_t, \tag{D.15}$$

$$\widehat{\widetilde{\Xi}}_t = -\widehat{\widetilde{C}}_t - \frac{2\delta_1 V^*}{1+2\delta_1 V^* - 2\sqrt{\delta_1\delta_2}}\widehat{V}_t, \tag{D.16}$$

$$\widehat{V}_t = \frac{1}{2\{(1+\delta_2)R_n^* - 1\}}\left(\widehat{Q}_t + \widehat{R}_{n,t}\right), \tag{D.17}$$

$$\widehat{V}_t = \widehat{\widetilde{C}}_t - \widehat{\widetilde{M_t/P_t}}, \tag{D.18}$$

$$\widehat{\widetilde{W_t/P_t}} = \max\left\{\widehat{\chi}_t - \widehat{\widetilde{\Xi}}_t + \frac{1}{\eta}\widehat{H}_t\ ,\ \ln\gamma + \widehat{\widetilde{W_{t-1}/P_{t-1}}} - (\widehat{\Pi}_t + \ln\Pi^*)\right\}, \tag{D.19}$$

$$\widehat{MC}_t = \widehat{\widetilde{W_t/P_t}} - \widehat{Z}_t, \tag{D.20}$$

$$\widehat{B_t/P_t} = \widehat{\Omega}_{1t} - \widehat{\Omega}_{2t}, \tag{D.21}$$

$$\begin{aligned}\widehat{\Omega}_{1t} &= \left(1-\lambda\beta(\Pi^*)^\theta\right)\left(\widehat{MC}_t + \widehat{\widetilde{\Xi}}_t + \widehat{\widetilde{Y}}_t\right)\\ &\quad + \left(\lambda\beta(\Pi^*)^\theta\right)\mathbb{E}_t\left[\theta\widehat{\Pi}_{t+1} + \widehat{\Omega}_{1t+1}\right],\end{aligned} \tag{D.22}$$

$$\begin{aligned}\widehat{\Omega}_{2t} &= \left(1-\lambda\beta(\Pi^*)^{\theta-1}\right)\left(\widehat{\widetilde{\Xi}}_t + \widehat{\widetilde{Y}}_t\right)\\ &\quad + \left(\lambda\beta(\Pi^*)^{\theta-1}\right)\mathbb{E}_t\left[(\theta-1)\widehat{\Pi}_{t+1} + \widehat{\Omega}_{2t+1}\right],\end{aligned} \tag{D.23}$$

$$0 = \left(1 - \lambda(\Pi^*)^{\theta-1}\right) \widehat{B_t/P_t} - \left(\lambda(\Pi^*)^{\theta-1}\right) \widehat{\Pi}_t, \tag{D.24}$$

$$\widehat{\widetilde{Y}}_t = \widehat{Z}_t + \widehat{H}_t - \widehat{D}_t, \tag{D.25}$$

$$\widehat{D}_t = \left(\lambda(\Pi^*)^{\theta}\right) \widehat{D}_{t-1} + \frac{\theta\lambda\left((\Pi^*)^{\theta} - (\Pi^*)^{\theta-1}\right)}{1 - \lambda(\Pi^*)^{\theta-1}} \widehat{\Pi}_t, \tag{D.26}$$

$$\widehat{\widetilde{Y}}_t = \widehat{\widetilde{C}}_t + \frac{\delta_1 V^* - \delta_2/V^*}{1 + 2\delta_1 V^* + \delta_2/V^* - 2\sqrt{\delta_1\delta_2}} \widehat{V}_t, \tag{D.27}$$

$$\widehat{\widetilde{Y}}_t^f = \widehat{Z}_t - \frac{1}{1+1/\eta} \widehat{\chi}_t, \tag{D.28}$$

where variables with hats denote the log-deviations from the deterministic steady state, i.e., $\widehat{X}_t = \ln X_t - \ln X^*$.

As for monetary policy rule, the Taylor rule is given by

$$\widehat{R}_{n,t}^d = \rho_r \widehat{R}_{n,t}^d + (1-\rho_r)\left(\phi_\pi \widehat{\Pi}_t + \phi_y(\widehat{\widetilde{Y}}_t - \widehat{\widetilde{Y}}_t^f)\right), \tag{D.29}$$

$$\widehat{R}_{n,t} = \max\{\widehat{R}_{n,t}^d\ ,\ 0\}, \tag{D.30}$$

whereas the RW rule is written as

$$\widehat{R}_{n,t}^b = \phi_\pi \widehat{\Pi}_t + \phi_y(\widehat{\widetilde{Y}}_t - \widehat{\widetilde{Y}}_t^f), \tag{D.31}$$

$$\widehat{\Gamma}_t = \widehat{\Gamma}_{t-1} + \widehat{R}_{n,t-1} - \widehat{R}_{n,t-1}^b, \tag{D.32}$$

$$\widehat{R}_{n,t} = \max\{\widehat{R}_{n,t}^b - \widehat{\Gamma}_t\ ,\ 0\}. \tag{D.33}$$

Notice that DNWR in (D.19) and the ZLB in (D.30) and (D.33) are dealt with in a piecewise fashion.

Lastly, exogenous processes are given by

$$\widehat{Z}_t = \rho_z \widehat{Z}_{t-1} + \epsilon_t^z\ ,\ \epsilon_t^z \sim i.i.d.N(0, \sigma_z^2), \tag{D.34}$$

$$\widehat{\chi}_t = \rho_\chi \widehat{\chi}_{t-1} + \epsilon_t^\chi\ ,\ \epsilon_t^\chi \sim i.i.d.N(0, \sigma_\chi^2), \tag{D.35}$$

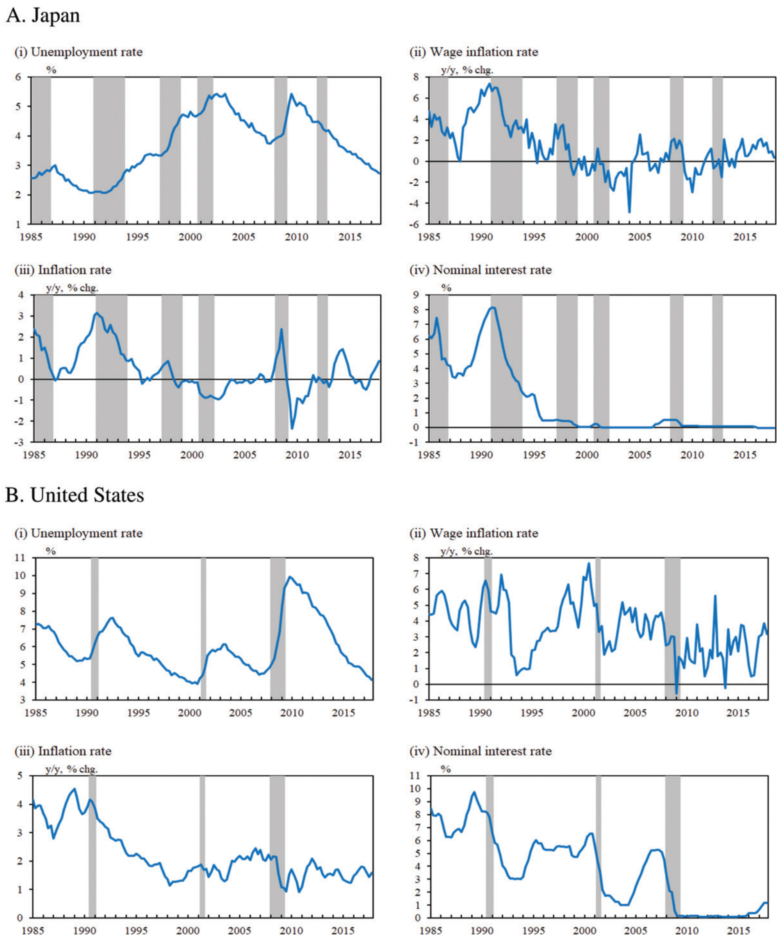
$$\widehat{Q}_t^{ar} = \rho_q \widehat{Q}_{t-1}^{ar} + \epsilon_t^q\ ,\ \epsilon_t^q \sim i.i.d.N(0, \sigma_q^2), \tag{D.36}$$

$$\widehat{Q}_t^{rs} = \begin{cases} -\frac{p_{21}}{p_{12}+p_{21}}\Delta \\ \frac{p_{12}}{p_{12}+p_{21}}\Delta \end{cases}, \tag{D.37}$$

$$\widehat{Q}_t = \widehat{Q}_t^{rs} + \widehat{Q}_t^{ar}. \tag{D.38}$$

Appendix E. Data

Figure E.1. Evolution of Key Variables



Note: The unemployment rate is obtained from the Ministry of Internal Affairs and Communications in Japan and from the Bureau of Labor Statistics in the United States. Definition and sources of other series are presented in the note for Table 2. The shaded areas indicate recessions.

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